SYNCHRONIZATION OF NETWORKED OSCILLATORS

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PHASE SYNCHRONIZATION: KURAMOTO MODEL (Kuramoto, 1984)

N "planar rotors", with phase ϕ_i and natural frequency ω_i: $\frac{\mathrm{d}\phi_i}{\mathrm{d}t} = \omega_i$

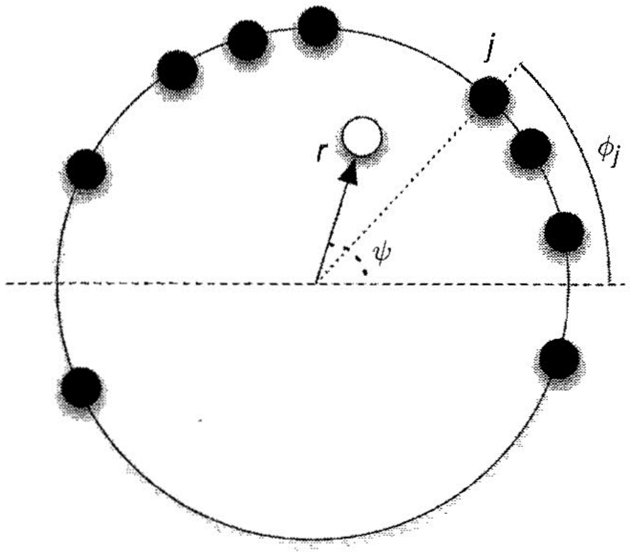
ω_i-s are random, extracted from a unimodal distribution with mean Ω

Nonlinear coupling ($V(i)$ = neighbours of i):

$$\frac{\mathrm{d}\phi_i}{\mathrm{d}t} = \omega_i + K \sum_{j \in V(i)} \sin\left(\phi_j - \phi_i\right)$$

Synchronization can be quantified by the "order parameter" $r(t)$ (centre of mass of the oscillators):

$$r(t)\mathrm{e}^{\mathrm{i}\psi(t)} = \frac{1}{N} \sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\phi_j(t)}$$

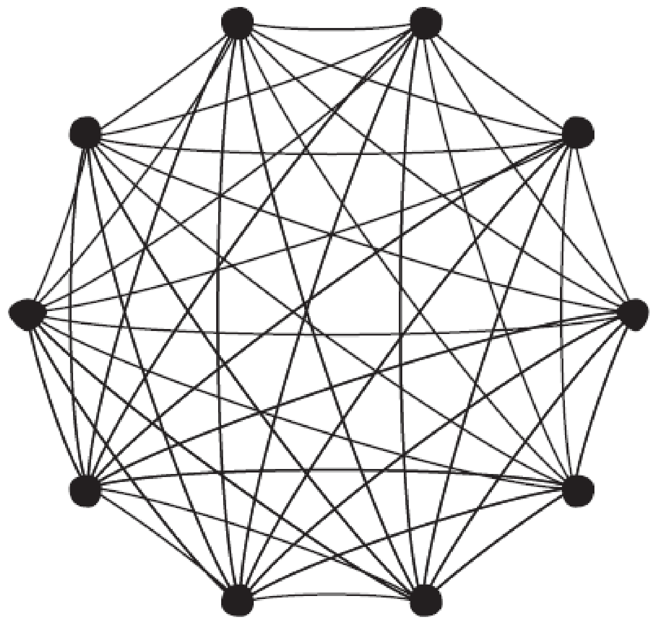


- $r(t) \to 0$: no synchronization
- $r(t) \to 1$: synchronization of all rotors (=identical $\mathrm{d}\phi_i/\mathrm{d}t$-s)
- If $r(t)$ tends to a nonzero value, a fraction of oscillators is synchronized.

Mean-field Kuramoto model

- N oscillators on a complete network
- coupling strength $K = K^0/N$
- combining $\frac{d\phi_i}{dt} = \omega_i + \frac{K^0}{N}\sum_j \sin(\phi_j - \phi_i)$ with $r(t)e^{i\psi(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{i\phi_j(t)}$ we obtain:

$$\frac{d\phi_i}{dt} = \omega_i + K^0 r \sin(\psi - \phi_i)$$



The coupling term depends on the "mean phase" ψ.

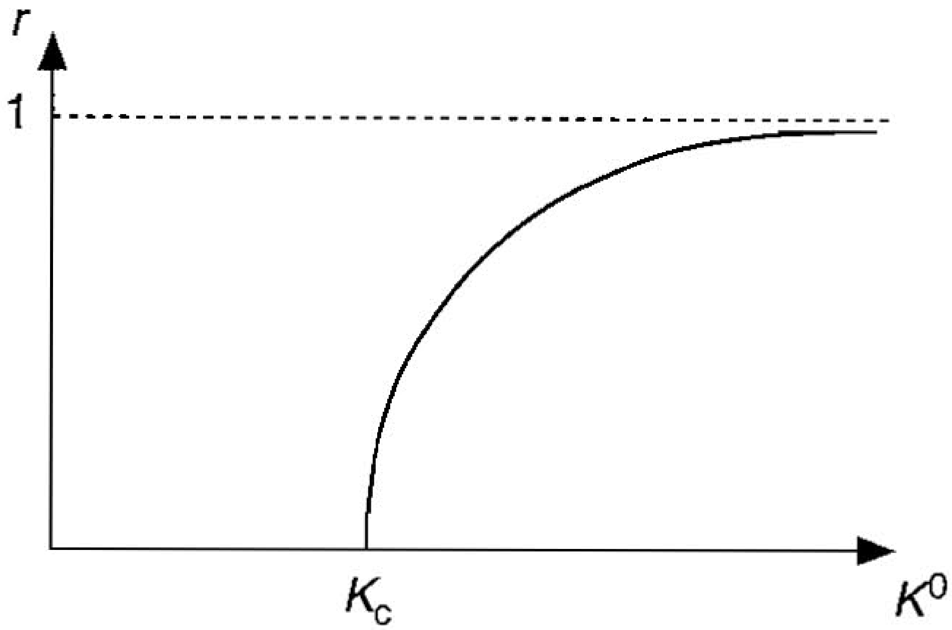
The coupling strength $K^0 r$ increases with r: positive feedback
(the largest the number of synchronized oscillators, the largest the chance to capture the remaining ones)

Numerical + analytical results: existence of a threshold value K_C

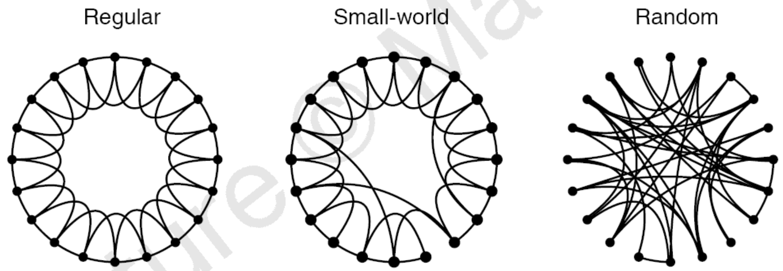
- $K^0 < K_C$: no synchronization ($r(t) \to 0$)

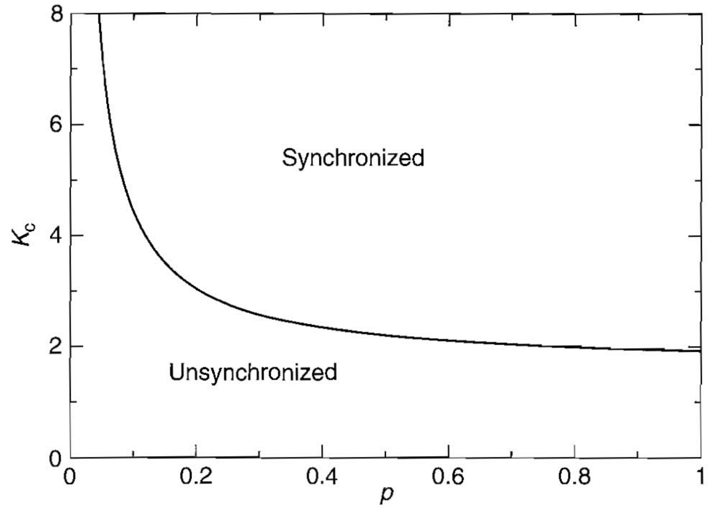
- $K^0 > K_C$: synchronization of larger and larger fractions of oscillators ($r(t) \to 1$ as $K^0 \to \infty$)

- when $r(t) \to r < 1$, the de-synchronized oscillators are those with largest detuning $|\omega_i - \Omega|$.



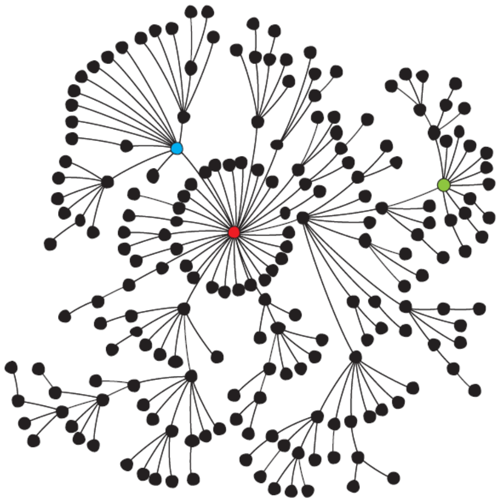
Kuramoto model on complex networks





Regular lattices: $K_C \to \infty$ as $N \to \infty$ (no synchronization)

Small-world networks (p = prob. of rewiring): K_C decreases with p



Heterogeneous networks (e.g., scale-free): $K_C \approx \frac{<k>}{<k^2>}$

$K_C \to 0$ in the limit $N \to \infty$: hubs drive the dynamics and impose synchronization

COMPLETE SYNCHRONIZATION IN NETWORKED OSCILLATORS

Isolated nodes i $(i = 1,2,\ldots,N)$ represent identical, autonomous, n-dimensional dynamical systems

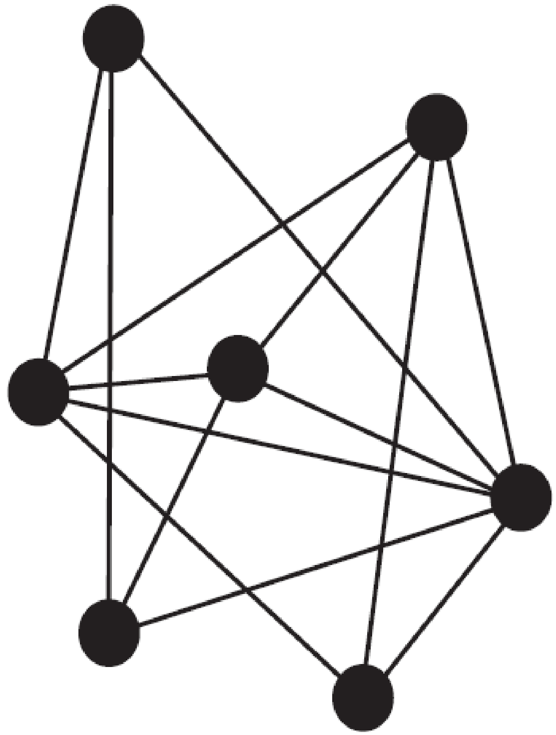
$$\dot{x} = f(x) \quad , \; x \in R^n$$

Their behaviour (when isolated) is oscillatory (periodic or chaotic).

The coupling is linear (diffusive):

$$\dot{x}^{(i)} = f(x^{(i)}) + \sum_{j:a_{ij}=1} d\left(H(x^{(j)} - x^{(i)})\right)$$

- $d \geq 0$ is the coupling strength,
- H is a $n \times n$ nonnegative matrix (diffusion profile) specifying which variables interact



Example: Each node represents a geographic location (island, patch) which is the habitat of a three-trophic food chain (resource R, consumer C, predator P).

The isolated demographic dynamics are described by the classical Rosenzweig-MacArthur model:

$$\dot{R} = rR\left(1 - \frac{R}{k}\right) - \frac{a_1 R}{1 + a_1 b_1 R} C$$

$$\dot{C} = e_1 \frac{a_1 R}{1 + a_1 b_1 R} C - d_1 C - \frac{a_2 C}{1 + a_2 b_2 C} P$$

$$\dot{P} = e_2 \frac{a_2 C}{1 + a_2 b_2 C} P - d_2 P$$

In this case H is diagonal: it specifies which are the species that disperse and sets the relative dispersal rates, e.g:

- only R disperses (e.g., seeds transported by the wind): $H = H' = diag\begin{bmatrix}1 & 0 & 0\end{bmatrix}$
- only C disperses (e.g., herbivores): $H = H'' = diag\begin{bmatrix}0 & 1 & 0\end{bmatrix}$
- only P disperses (e.g., carnivores): $H = H''' = diag\begin{bmatrix}0 & 0 & 1\end{bmatrix}$
- all variables disperse, at different rates: $H = H'''' = diag\begin{bmatrix}1 & 0.1 & 0.01\end{bmatrix}$

The overall dynamics are governed by the $N \times n$ equations:

$$\dot{x}^{(i)} = f(x^{(i)}) - d \sum_{j=1,2,\ldots,N} l_{ij} H x^{(j)} \quad , \quad i = 1,2,\ldots,N$$

where

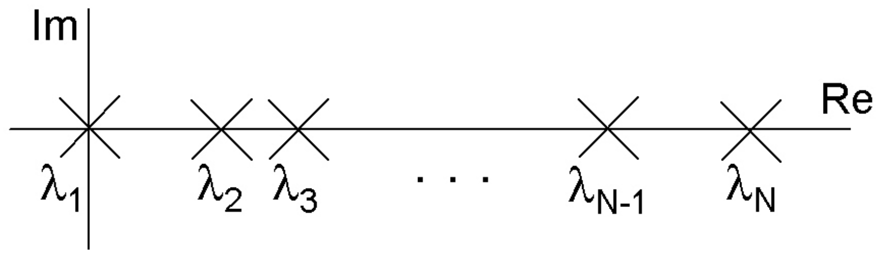
- if $i \neq j$, $l_{ij} = l_{ji} = -a_{ij} = -a_{ji}$ (=-1 if the link $i \leftrightarrow j$ exists, 0 otherwise)
- if $i = j$, $l_{ii} = -\Sigma_{j \neq i} l_{ij}$ (= degree of i)

$L = [l_{ij}]$ is the Laplacian matrix of the undirected network:

- real and symmetric (thus diagonalizable)
- all the off-diagonal entries are non-positive
- all rows have zero sum
- irreducible if the network is connected

It follows that the spectrum of L has the form

$$\sigma(L) = \{0 = \lambda_1 < \lambda_2 \ldots \leq \lambda_N\}$$



Complete synchronization

We have complete synchronization when

$$x^{(1)}(t) = x^{(2)}(t) = \ldots = x^{(N)}(t) = \bar{x}(t) \quad \forall t$$

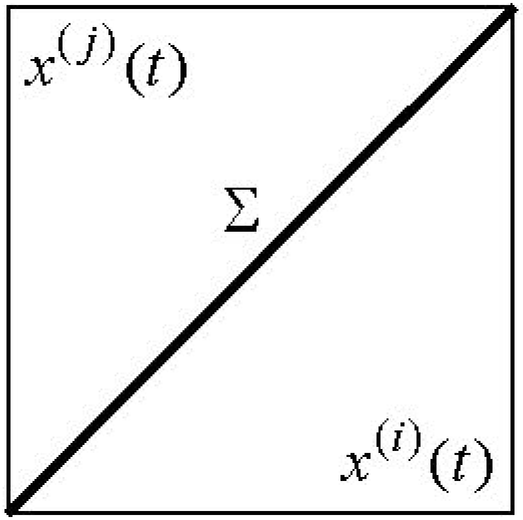
Since

$$\dot{x}^{(i)} = f(x^{(i)}) + \sum_{j:a_{ij}=1} d\Big(H(x^{(j)} - x^{(i)})\Big),$$

at synchronization all interaction terms $dH(x^{(j)} - x^{(i)})$ vanish: $\bar{x}(t)$ must be a solution of the isolated system $\dot{x} = f(x)$.

The synchronized trajectory lies in n-dimensional subspace Σ defined by

$$x^{(i)}(t) = \bar{x}(t),\ i = 1,2,\ldots,N$$

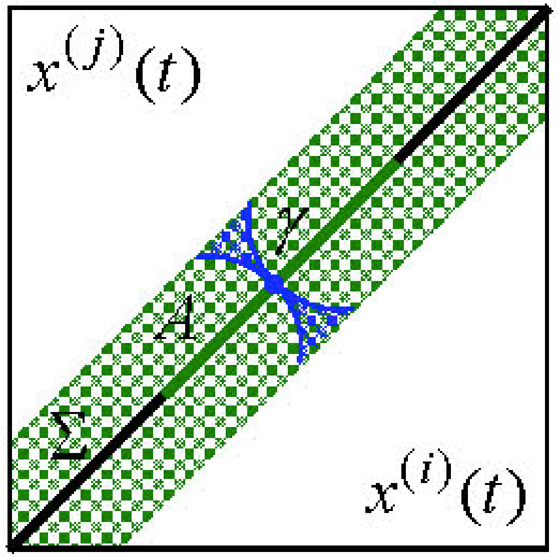


The synchronized solution $\bar{x}(t)$ has $N \times n$ Liapunov exponents:

- n are the exponents of the (periodic or chaotic) attractor of $\dot{x} = f(x)$ (perturbations within Σ)
- the remaining $(N-1) \times n$ are related to the perturbations transversal to Σ

The stability of the synchronized solution $\bar{x}(t)$ requires that all the $(N-1) \times n$ transversal Liapunov exponents be negative.

Technically, the above condition guarantees that the synchronized solution has a basin of attraction with positive measure (volume).



Master stability equation - Master stability function

Since L is diagonalizable, the variational equation (linearization) around $x^{(i)}(t) = \bar{x}(t)$ can be decomposed in N independent blocks, each one related to one single eigenvalue λ_i of L:

$$\dot{v}_i = J(\bar{x}(t))v_i - d\lambda_i H v_i \quad , \quad i = 1,2,\ldots,N$$

$J(x) = \partial f / \partial x$ is the Jacobian of $f(x)$.

- For $i = 1$ ($\lambda_1 = 0$):

  $\dot{v}_i = J(\bar{x}(t))v_i$ relates to the dynamics within the synchronization subspace Σ

- For $i = 2,3,\ldots,N$ ($\lambda_i > 0$):

  $\dot{v}_i = J(\bar{x}(t))v_i - d\lambda_i H v_i$ relates to the dynamics within the i-th direction transversal to Σ

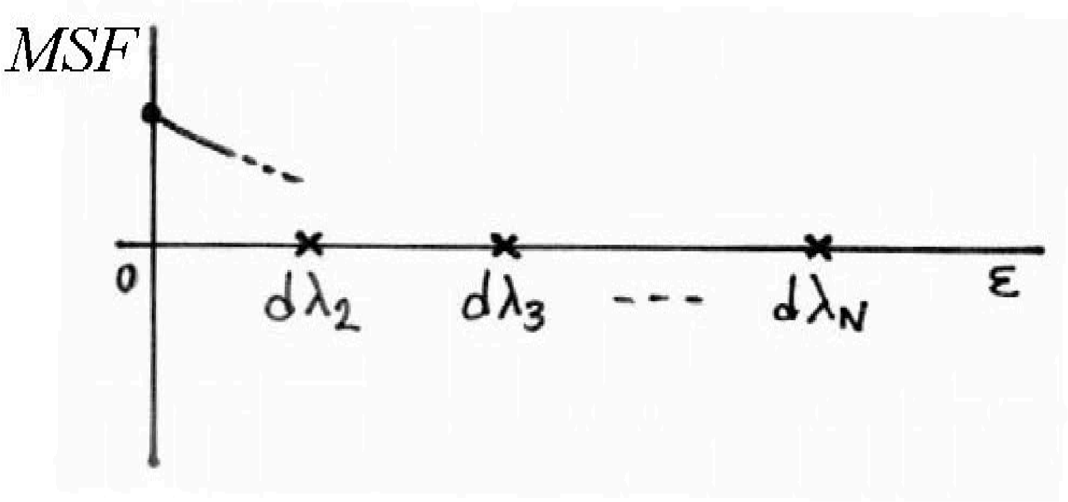
The stability of $\bar{x}(t)$ requires that, for each $i = 2,3,\ldots,N$, the maximal Liapunov exponent of the latter equation be negative.

Instead of the $N-1$ equations (n-dim) related to $0<\lambda_2 \ldots \leq \lambda_N$, we discuss one single parameterized equation (n-dim) (Master Stability Equation – MSE):

$$\dot{v} = J(\bar{x}(t))v - \varepsilon H v, \quad \text{where } \varepsilon = d\lambda \geq 0$$

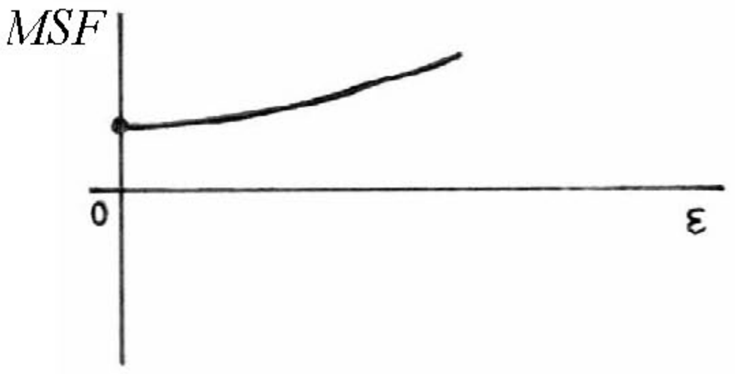
If MSF is the maximal Liapunov exponent of the MSE for a given ε, then the function $MSF = MSF(\varepsilon)$ is the Master Stability Function – MSF.

Synchronization requires $MSF(\varepsilon) < 0$ for all $\varepsilon = d\lambda_i$.



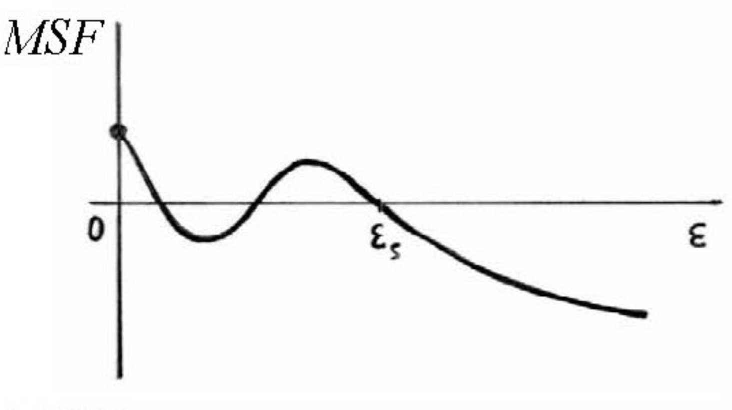
Remark: $MSF(0) > 0$ $(=0)$ if the solution $\bar{x}(t)$ of the isolated system $\dot{x} = f(x)$ is chaotic (periodic).

Remark: the MSF depends on the system (f) and diffusion profile (H), but NOT on the network topology (L).



MSF type I: $MSF(\varepsilon) > 0$ for all $\varepsilon > 0$

synchronization is impossible on all networks



MSF type II: $MSF(\varepsilon) < 0$ for all $\varepsilon > \varepsilon_s > 0$

synchronization is possible on all networks, provided

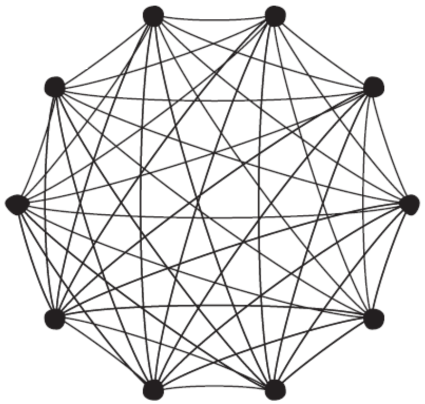
$$d > \frac{\varepsilon_s}{\lambda_2}$$

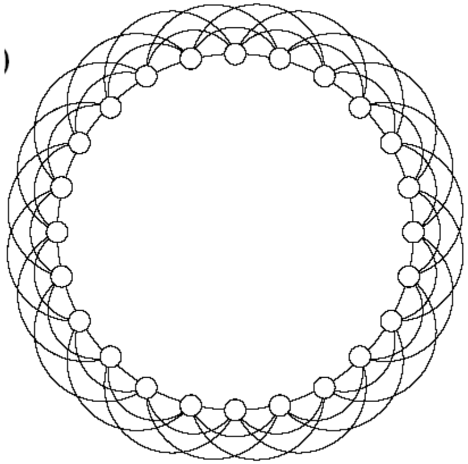
Given a MSF type II, synchronization is favoured in networks with large λ_2, because $\varepsilon_s / \lambda_2$ is small (=less coupling strength d needed).

Complete network:

$$\lambda_2 = N$$

Synchronization is favoured as N grows.



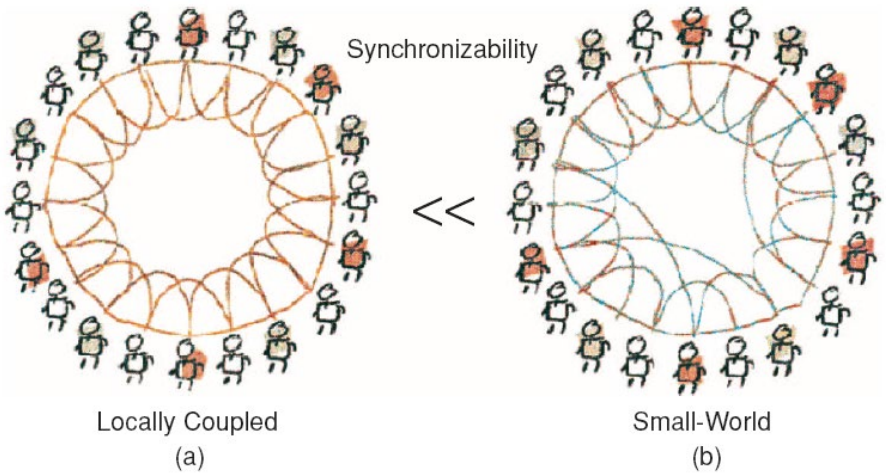


Watts-Strogatz "loop":

$$\lambda_2 \to 0 \text{ per } N \to \infty$$

Synchronization is more difficult as N grows.

In a Watts-Strogatz loop, synchronization can be greatly favoured by adding a few "long distance" connections (small-world network).



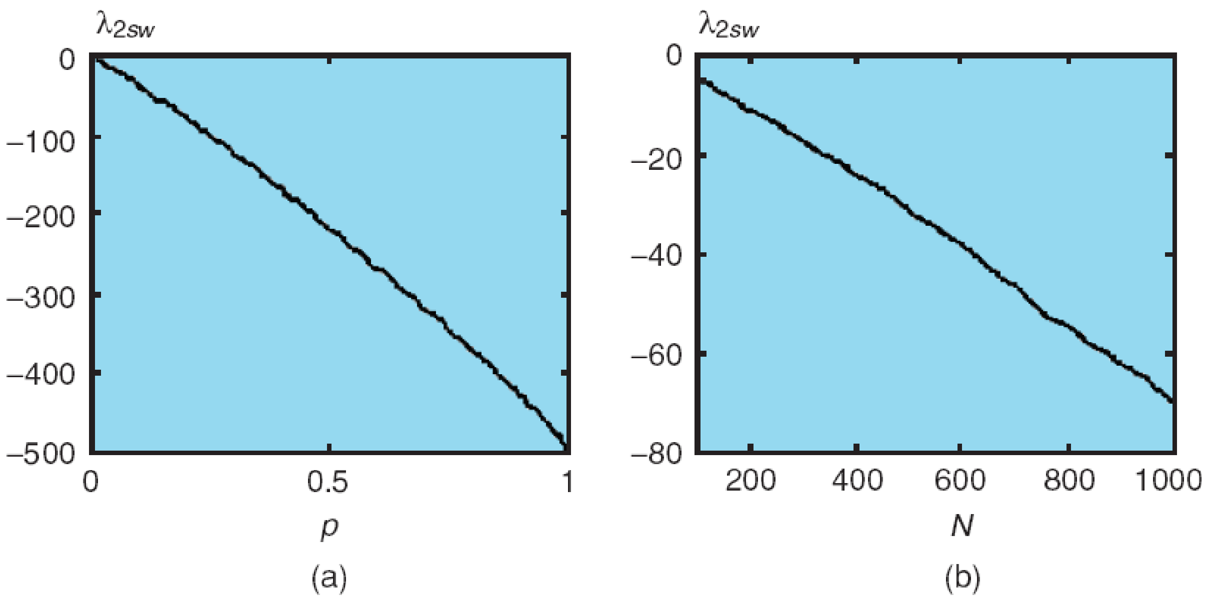
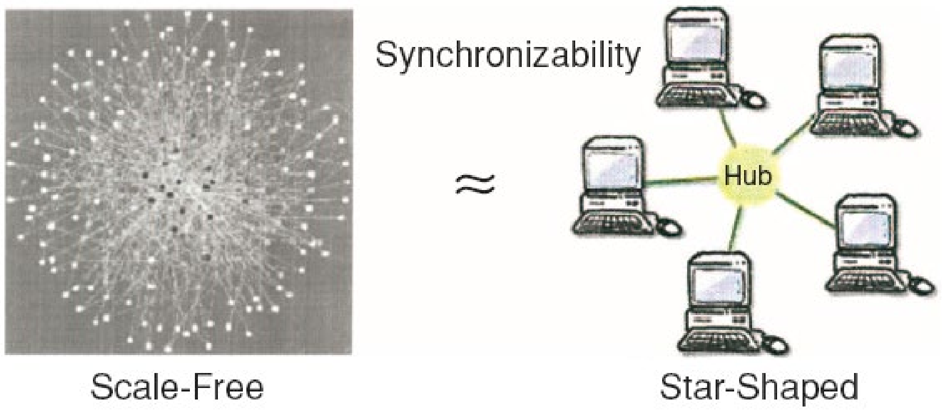


Figure 14. The second-largest eigenvalue λ_{2sw} of the coupling matrix of the small-world network (1) [41]. (a) λ_{2sw} as a function of the adding probability p with the network size $N = 500$. (b) λ_{2sw} as a function of the network size with adding probability $p = 0.1$.

In a scale-free network λ_2 is close to 1, similarly to a pure star network ($\lambda_2 = 1$).



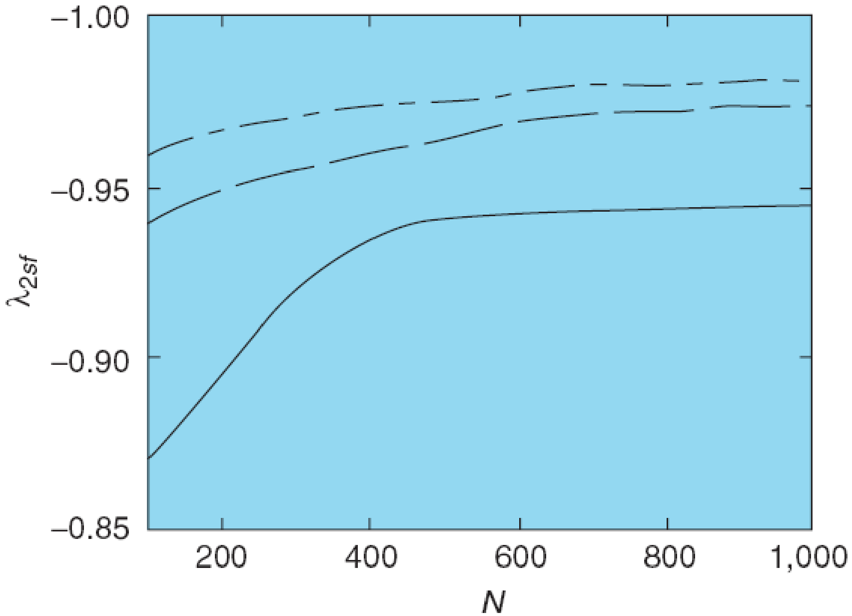
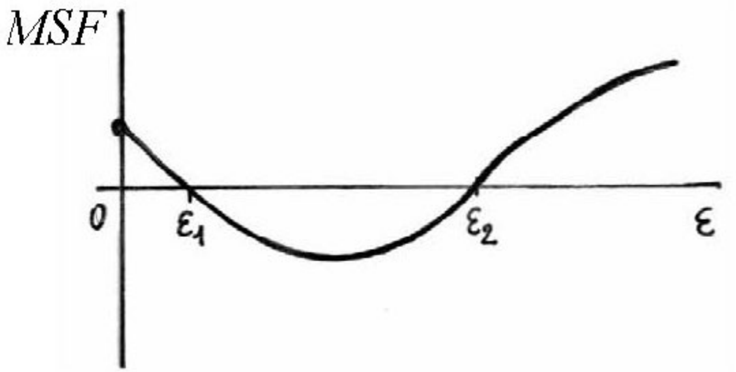


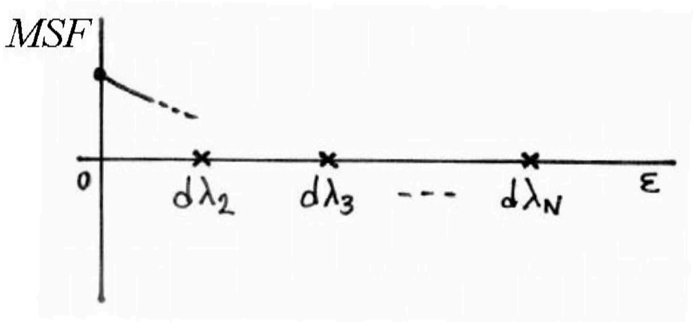
Figure 16. The second-largest eigenvalue of the coupling matrix of the scale-free network (1), for $m_0 = m = 3$ (−); $m_0 = m = 5$(— —); and $m_0 = m = 7$(—·) [42].



MSF type III:

synchronization is possible (with suitable d) if

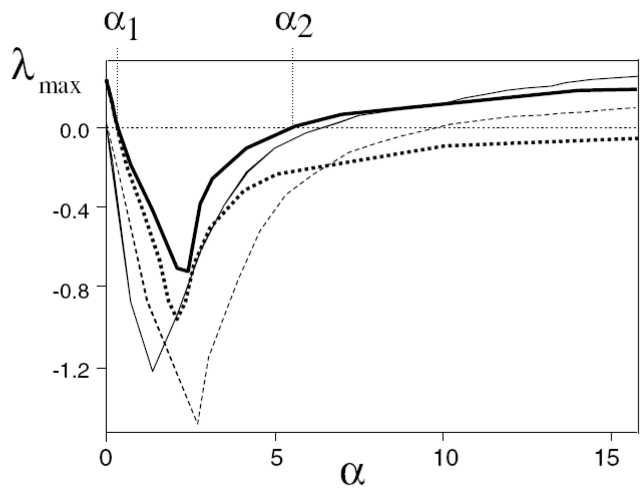
$$\frac{\lambda_N}{\lambda_2} < \frac{\varepsilon_2}{\varepsilon_1}$$



Remark: it is not true that any network can be synchronized provided d is sufficiently large:

- in some networks, when d grows we have first synchronization and then de-synchronization
- some networks cannot be synchronized by any d

FIG. 1. Four typical master stability functions for coupled Rössler oscillators: chaotic (bold) and periodic (regular lines); with y coupling (dashed) and x coupling (solid lines). (All scaled for clearer visualization.) We concentrate on the x-coupled chaotic case with a negative region (α_1, α_2).



Given a MSF type III, synchronization is favoured in networks with small λ_N / λ_2.

In a Watts-Strogatz "loop", λ_N / λ_2 diverges with N ($\lambda_N / \lambda_2 \cong \alpha N^2 / (m(m+1))$).

Again, synchronization can strongly be favoured (at constant N) by adding a few “long distance” connections (small-world network).

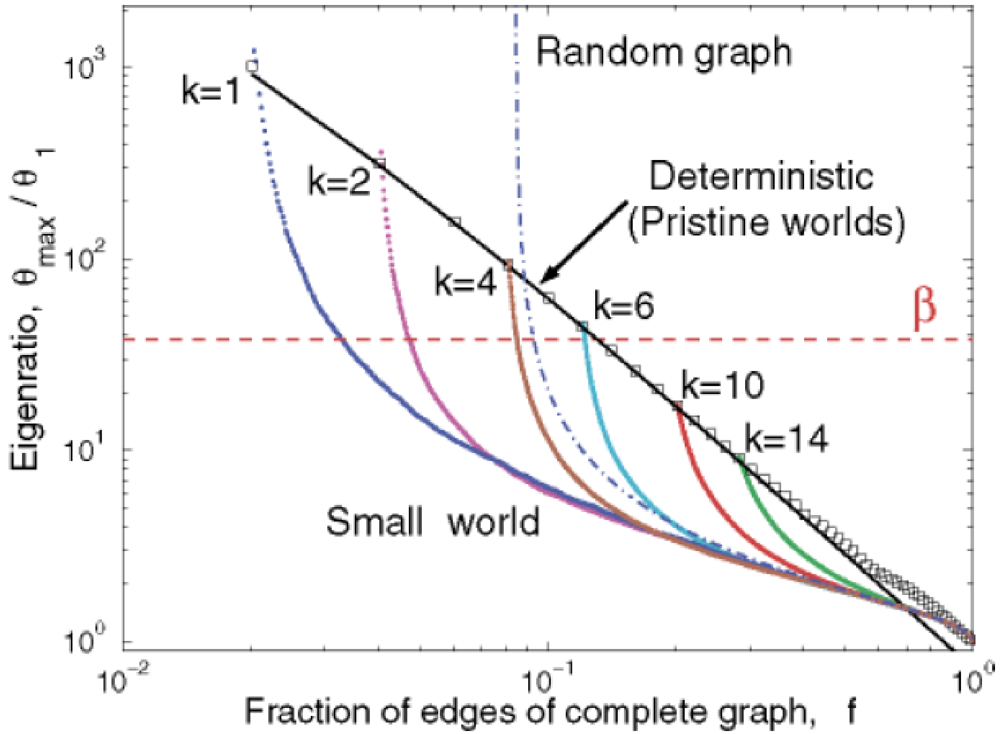
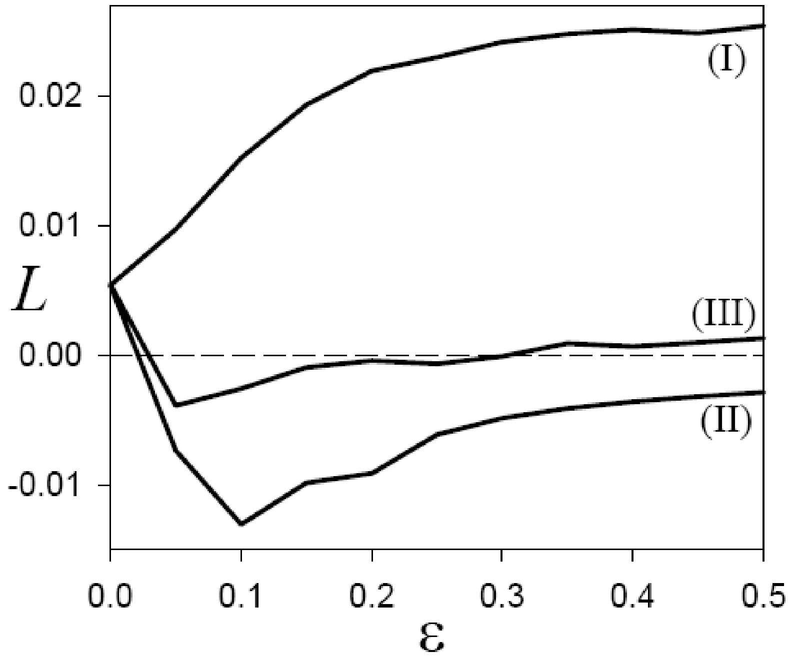


FIG. 2 (color online). Decay of the eigenratio in a $n = 100$ lattice as $f\binom{n}{2}$ edges are added following purely deterministic, semirandom (SW), and purely random schemes. Networks become synchronizable below the dashed line (β). The squares (numerical) and the solid line [analytic Eq. (4)] show the eigenratio decay of pristine worlds through the deterministic addition of short-range connections. The dot-dashed line corresponds to purely random graphs [Eq. (5)], which become *almost surely* disconnected and unsynchronizable at $f \simeq 0.0843$. The semirandom SW approach (dots, shown for ranges $k = 1, 2, 4, 6, 10, 14$) is more efficient in producing synchronization when $k < \ln n$.

Example: three-trophic food chain



- (I) $H = H' = diag[1 \;\; 0 \;\; 0]$: synchronization is impossible for any network
- (II) $H = H'' = diag[0 \;\; 1 \;\; 0]$: synchronization is possible for any network, provided d be sufficiently large
- (III) $H = H''' = diag[0 \;\; 0 \;\; 1]$: synchronization is possible (with suitable d) only for some networks