COMPLETE SYNCHRONIZATION OF COUPLED OSCILLATORS

Carlo PICCARDI

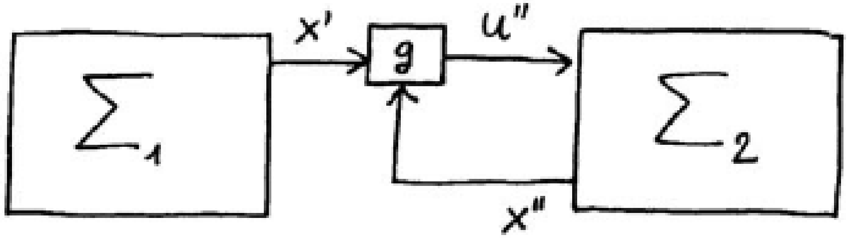
DEIB - Department of Electronics, Information and Bioengineering
Politecnico di Milano, Italy

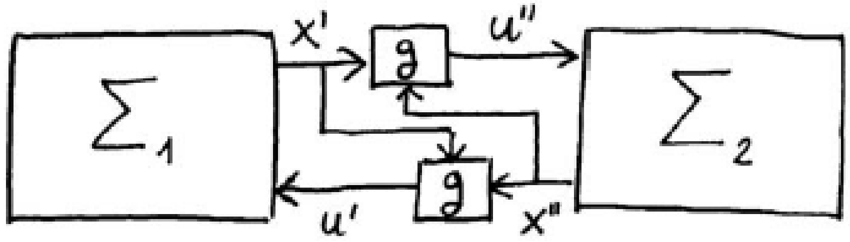
email carlo.piccardi@polimi.it
https://piccardi.faculty.polimi.it

COMPLETE SYNCHRONIZATION

Consider two systems Σ_1 e Σ_2:

- identical (same f): $\dot{x}' = f(x',u')$, $\dot{x}'' = f(x'',u'')$
- in oscillatory (periodic or chaotic) regime when isolated ($u'(t) = u''(t) = 0 \ \forall t$)
- interacting uni- or bi-directionally: $u' = g(x',x'')$, $u'' = g(x'',x')$





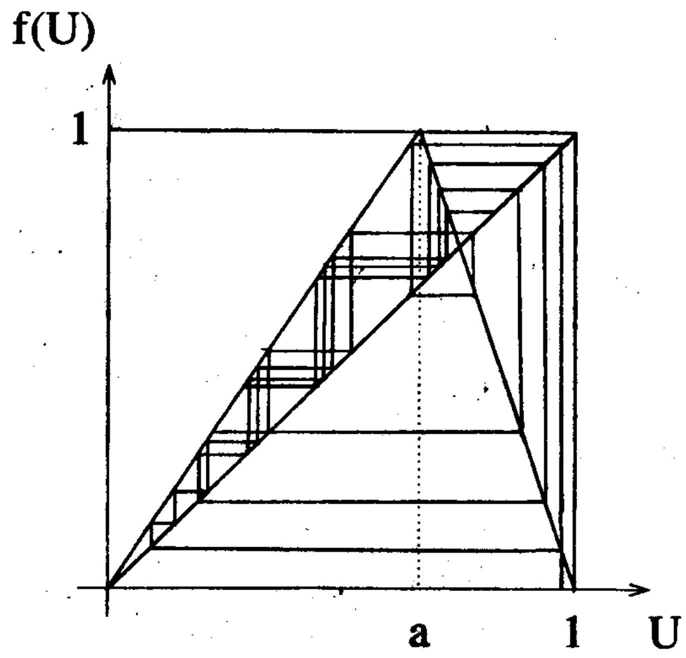
Σ_1 and Σ_2 are completely synchronized if

$$\lim_{t\to\infty}\left|x'(t)-x''(t)\right|=0$$

Remarks:

- The definition implies that the "synchronized state" $x'(t)=x''(t)$ be asymptotically stable (at least locally).
- Differently from phase synchronization (=same average frequency but amplitudes not necessarily correlated), complete synchronization implies the perfect coincidence of the behaviours of the two systems.
- For obtaining complete synchronization, the interaction might be non "weak".
- Complete synchronization preserves the periodic/chaotic behaviour.
- More in general, the two systems could be non-identical but "similar" (e.g., same state equations but slightly different parameters). The requirement is relaxed to $|x'(t)-x''(t)|<$ constant.

Example: synchronization of skewed tent maps



Consider two 1st-order discrete-time systems ("skewed" tent maps, $a = 0.7$):

$$f(U) = \begin{cases} U/a & \text{se } 0 \le U < a \\ (1-U)/(1-a) & \text{se } a \le U \le 1 \end{cases}$$

Both systems (isolated) have chaotic dynamics.

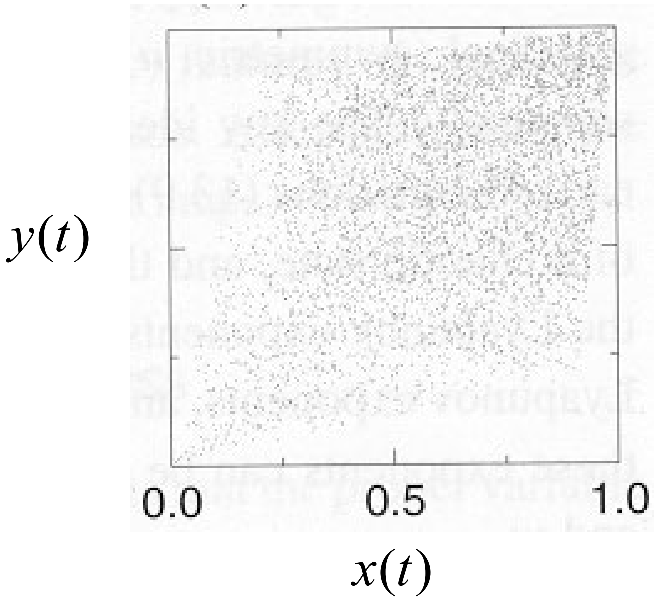
The two systems are coupled bi-directionally:

$$\begin{aligned} x(t+1) &= (1-\varepsilon) f(x(t)) + \varepsilon f(y(t)) \\ y(t+1) &= \varepsilon f(x(t)) + (1-\varepsilon) f(y(t)) \end{aligned}$$

The parameter ε is the coupling strength:

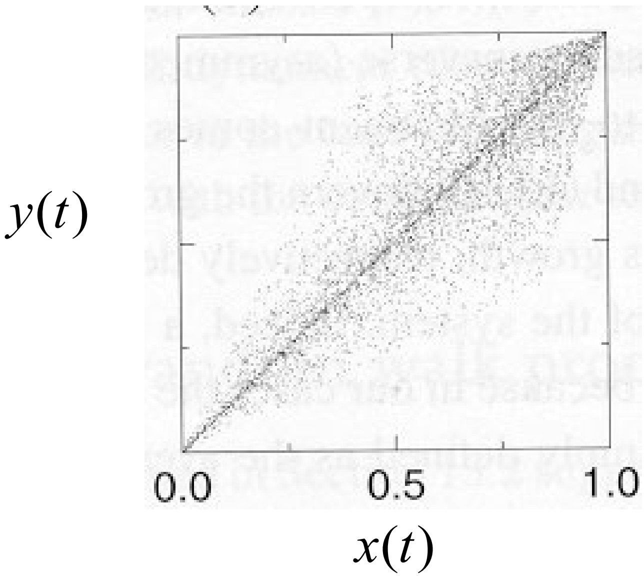
$\varepsilon = 0$ (no interaction)

$x(t)$ and $y(t)$ evolve independently in chaotic regime.



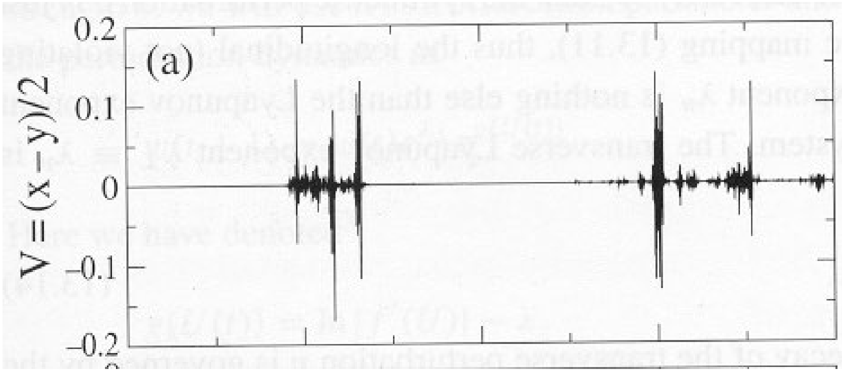
$\varepsilon = 0.2$ (weak interaction)

$x(t)$ and $y(t)$ show the tendency to synchronize.



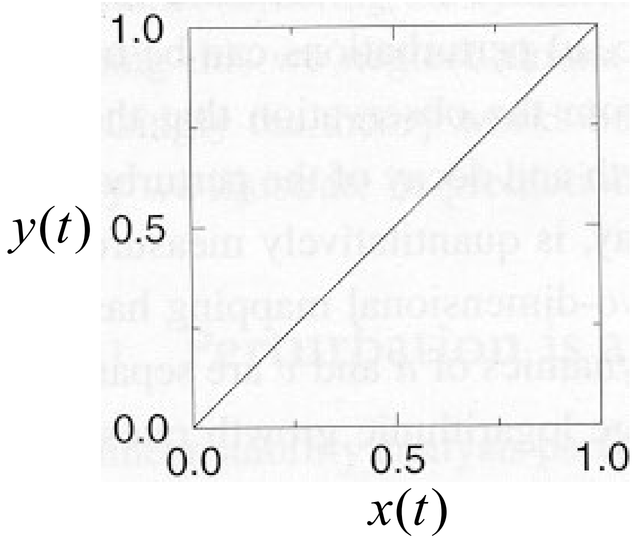
By numerical analysis: complete synchronization takes place for $\varepsilon > \varepsilon_c \cong 0.228$.

With ε slightly smaller than ε_c (e.g. $\varepsilon = \varepsilon_c - 0.001$) we detect intervals of apparent synchronization, interrupted by burst of de-synchronization ("intermittencies").



$\varepsilon = 0.3$ (complete synchronization)

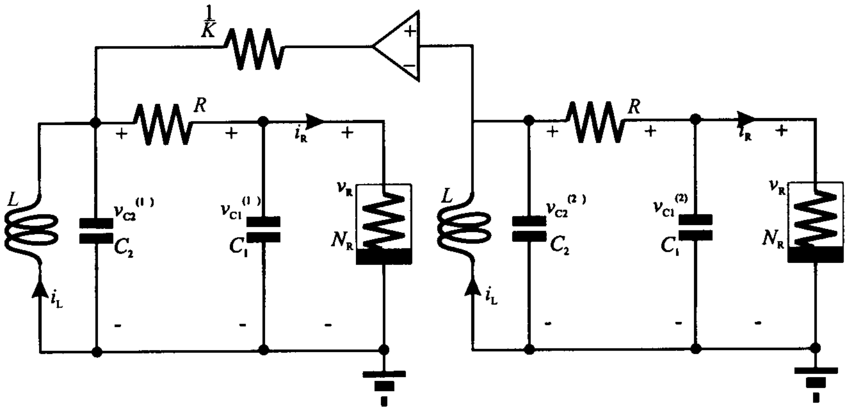
$x(t)$ and $y(t)$ coincide in all time instants and fill the interval (0,1).



$\varepsilon = 1/2$ gives the maximal interaction: we have $x(t) = y(t)$ from $t = 1$ (check the equations!) for all initial conditions (complete synchronization in finite time).

Example: lab synchronization of two Chua circuits

Two circuits identical in theory (slightly different in practice, due to tolerances of the components) interact trough unidirectional coupling:

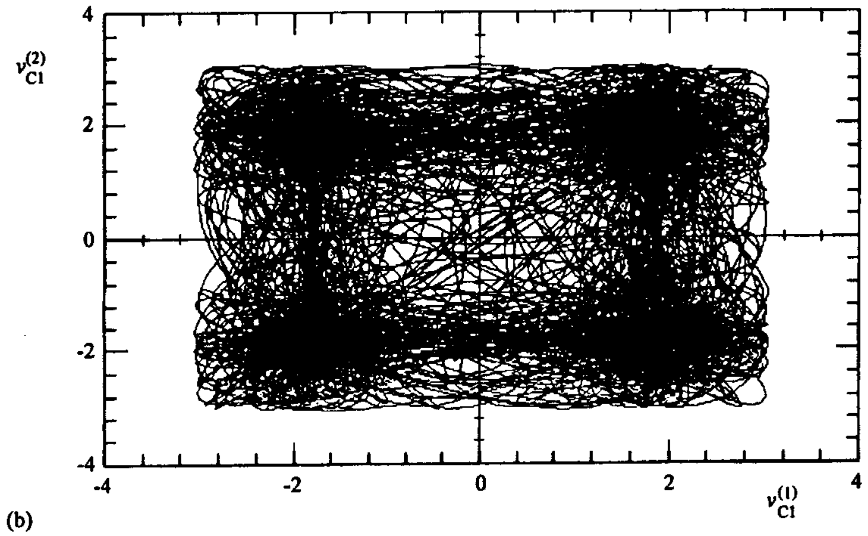


The (adimensional) equations of the two systems are:

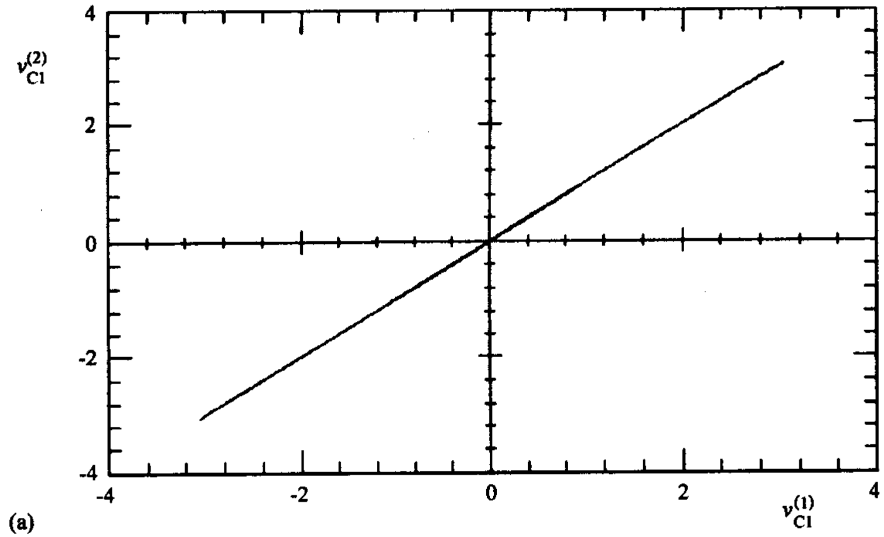
$$\begin{aligned} \dot{x}' &= \alpha(y' - x' - h(x')) \\ \dot{y}' &= x' - y' + z' + K(y'' - y') \\ \dot{z}' &= -\beta y' \end{aligned} \qquad \begin{aligned} \dot{x}'' &= \alpha(y'' - x'' - h(x'')) \\ \dot{y}'' &= x'' - y'' + z'' \\ \dot{z}'' &= -\beta y'' \end{aligned}$$

K is the coupling strength.

Complete synchronization takes place above the critical value $K = K_c \in (1.1, 1.2)$.



$K = 1.1$



$K = 1.2$

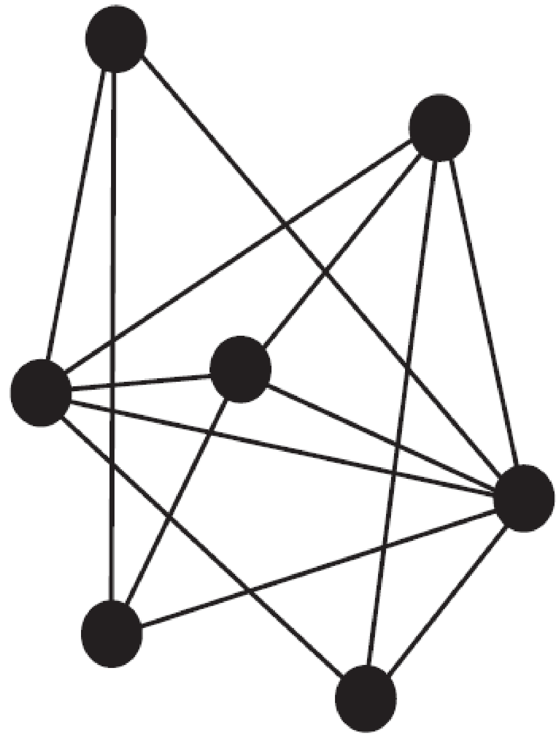
COMPLETE SYNCHRONIZATION IN NETWORKED OSCILLATORS

Isolated nodes i $(i = 1,2,\dots,N)$ represent identical, autonomous, n-dimensional dynamical systems

$$\dot{x} = f(x) \quad , \; x \in R^n$$

Their behaviour (when isolated) is oscillatory (periodic or chaotic).

The coupling is linear (diffusive):



$$\dot{x}^{(i)} = f(x^{(i)}) + \sum_{j:a_{ij}=1} d\left(H(x^{(j)} - x^{(i)})\right)$$

- $d \geq 0$ is the coupling strength,
- H is a $n \times n$ nonnegative matrix (diffusion profile) specifying which variables interact

Example: Each node represents a geographic location (island, patch) which is the habitat of a three-trophic food chain (resource R, consumer C, predator P).

The isolated demographic dynamics are described by the classical Rosenzweig-MacArthur model:

$$\dot{R} = rR\left(1 - \frac{R}{k}\right) - \frac{a_1 R}{1 + a_1 b_1 R} C$$

$$\dot{C} = e_1 \frac{a_1 R}{1 + a_1 b_1 R} C - d_1 C - \frac{a_2 C}{1 + a_2 b_2 C} P$$

$$\dot{P} = e_2 \frac{a_2 C}{1 + a_2 b_2 C} P - d_2 P$$

In this case H is diagonal: it specifies which are the species that disperse and sets the relative dispersal rates, e.g:

- only R disperses (e.g., seeds transported by the wind): $H = H' = diag\begin{bmatrix}1 & 0 & 0\end{bmatrix}$
- only C disperses (e.g., herbivores): $H = H'' = diag\begin{bmatrix}0 & 1 & 0\end{bmatrix}$
- only P disperses (e.g., carnivores): $H = H''' = diag\begin{bmatrix}0 & 0 & 1\end{bmatrix}$
- all variables disperse, at different rates: $H = H'''' = diag\begin{bmatrix}1 & 0.1 & 0.01\end{bmatrix}$

The overall dynamics are governed by the $N \times n$ equations:

$$\dot{x}^{(i)} = f(x^{(i)}) - d \sum_{j=1,2,\dots,N} l_{ij} H x^{(j)} \quad , \quad i = 1,2,\dots,N$$

where

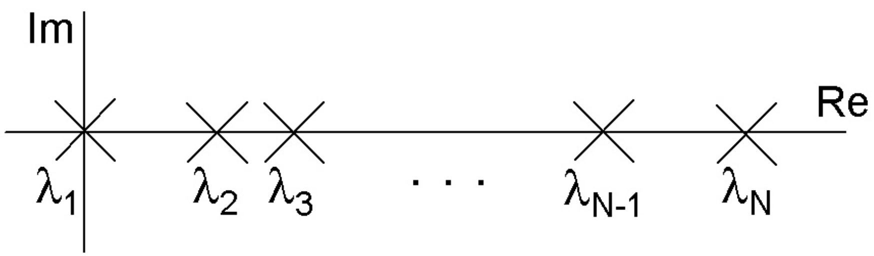
- if $i \neq j$, $l_{ij} = l_{ji} = -a_{ij} = -a_{ji}$ (=-1 if the link $i \leftrightarrow j$ exists, 0 otherwise)
- if $i = j$, $l_{ii} = -\Sigma_{j \neq i} l_{ij}$ (= degree of i)

$L = [l_{ij}]$ is the Laplacian matrix of the undirected network:

- real and symmetric (thus diagonalizable)
- all the off-diagonal entries are non-positive
- all rows have zero sum
- irreducible if the network is connected

It follows that the spectrum of L has the form

$$\sigma(L) = \{0 = \lambda_1 < \lambda_2 \ldots \leq \lambda_N\}$$



Complete synchronization

We have complete synchronization when

$$x^{(1)}(t) = x^{(2)}(t) = \ldots = x^{(N)}(t) = \bar{x}(t) \quad \forall t$$

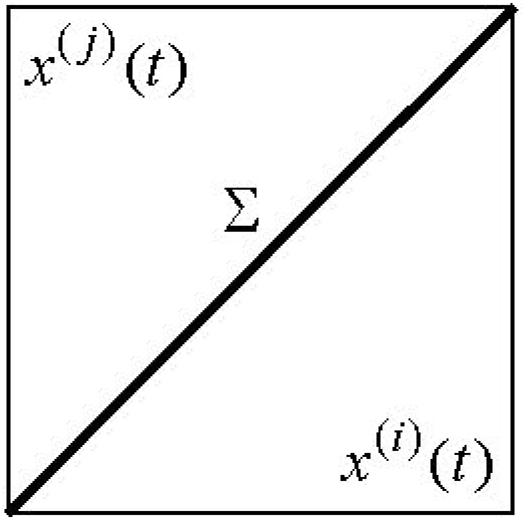
Since

$$\dot{x}^{(i)} = f(x^{(i)}) + \sum_{j:a_{ij}=1} d\Big(H(x^{(j)} - x^{(i)})\Big),$$

at synchronization all interaction terms $dH(x^{(j)} - x^{(i)})$ vanish: $\bar{x}(t)$ must be a solution of the isolated system $\dot{x} = f(x)$.

The synchronized trajectory lies in n-dimensional subspace Σ defined by

$$x^{(i)}(t) = \bar{x}(t),\; i = 1,2,\ldots,N$$

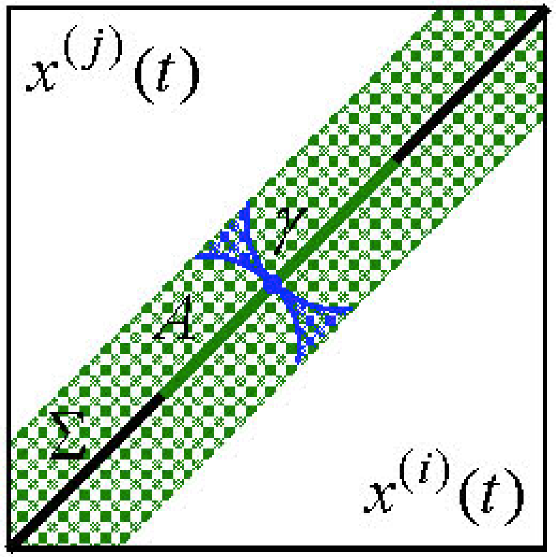


The synchronized solution $\bar{x}(t)$ has $N \times n$ Liapunov exponents:

- n are the exponents of the (periodic or chaotic) attractor of $\dot{x} = f(x)$ (perturbations within Σ)
- the remaining $(N-1)\times n$ are related to the perturbations transversal to Σ

The stability of the synchronized solution $\bar{x}(t)$ requires that all the $(N-1)\times n$ transversal Liapunov exponents be negative.

Technically, the above condition guarantees that the synchronized solution has a basin of attraction with positive measure (volume).



Master stability equation - Master stability function

Since L is diagonalizable, the variational equation (linearization) around $x^{(i)}(t)=\bar{x}(t)$ can be decomposed in N independent blocks, each one related to one single eigenvalue λ_i of L:

$$\dot{v}_i = J(\bar{x}(t))v_i - d\lambda_i H v_i \quad , \quad i=1,2,\dots,N$$

$J(x)=\partial f/\partial x$ is the Jacobian of $f(x)$.

- For $i=1$ ($\lambda_1=0$):

  $\dot{v}_i = J(\bar{x}(t))v_i$ relates to the dynamics within the synchronization subspace Σ

- For $i=2,3,\dots,N$ ($\lambda_i>0$):

  $\dot{v}_i = J(\bar{x}(t))v_i - d\lambda_i H v_i$ relates to the dynamics within the i-th direction transversal to Σ

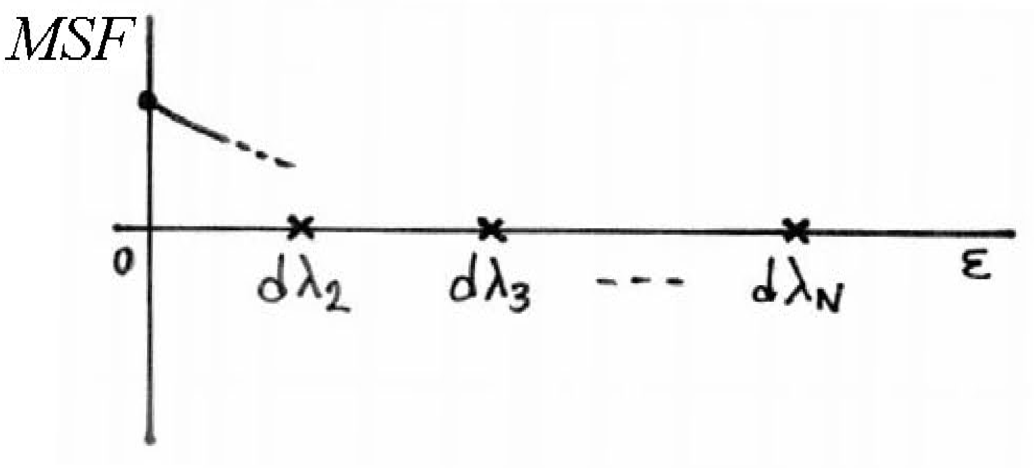
The stability of $\bar{x}(t)$ requires that, for each $i=2,3,\dots,N$, the maximal Liapunov exponent of the latter equation be negative.

Instead of the $N-1$ equations (n-dim) related to $0<\lambda_2 \ldots \leq \lambda_N$, we discuss one single parameterized equation (n-dim) (Master Stability Equation – MSE):

$$\dot{v} = J(\bar{x}(t))v - \varepsilon H v, \quad \text{where } \varepsilon = d\lambda \geq 0$$

If MSF is the maximal Liapunov exponent of the MSE for a given ε, then the function $MSF = MSF(\varepsilon)$ is the Master Stability Function – MSF.

Synchronization requires $MSF(\varepsilon) < 0$ for all $\varepsilon = d\lambda_i$.

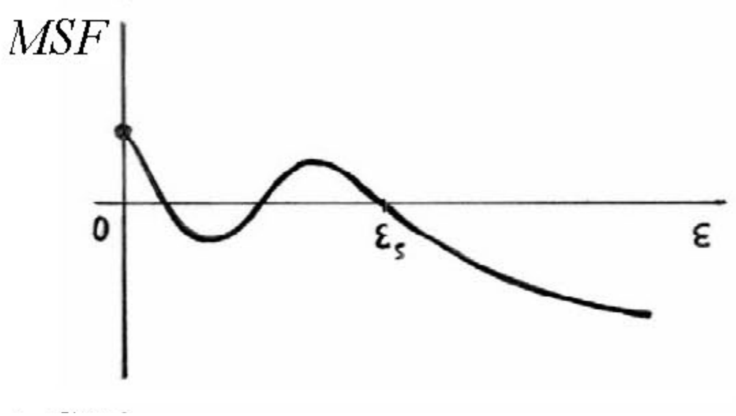


Remark: $MSF(0) > 0$ $(= 0)$ if the solution $\bar{x}(t)$ of the isolated system $\dot{x} = f(x)$ is chaotic (periodic).

Remark: the MSF depends on the system (f) and diffusion profile (H), but NOT on the network topology (L).

MSF type I: $MSF(\varepsilon) > 0$ for all $\varepsilon > 0$

synchronization is impossible on all networks



MSF type II: $MSF(\varepsilon) < 0$ for all $\varepsilon > \varepsilon_S > 0$

synchronization is possible on all networks, provided

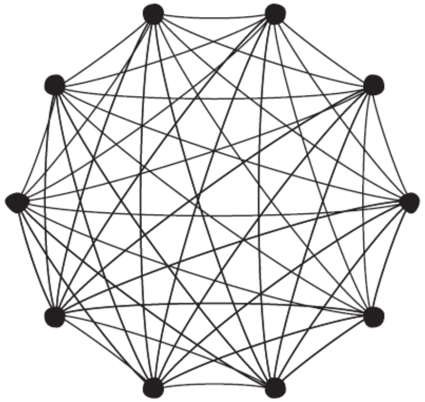
$$d > \frac{\varepsilon_S}{\lambda_2}$$

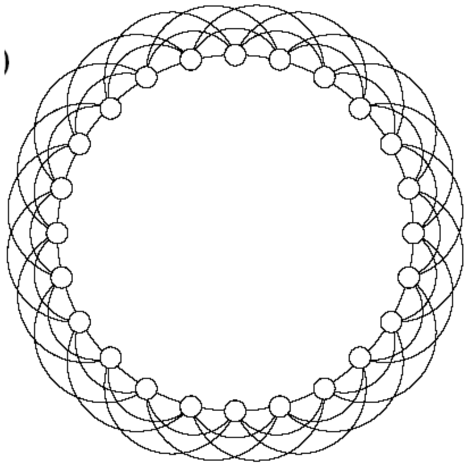
Given a MSF type II, synchronization is favoured in networks with large λ_2, because $\varepsilon_s / \lambda_2$ is small (=less coupling strength d needed).

Complete network:

$$\lambda_2 = N$$

Synchronization is favoured as N grows.



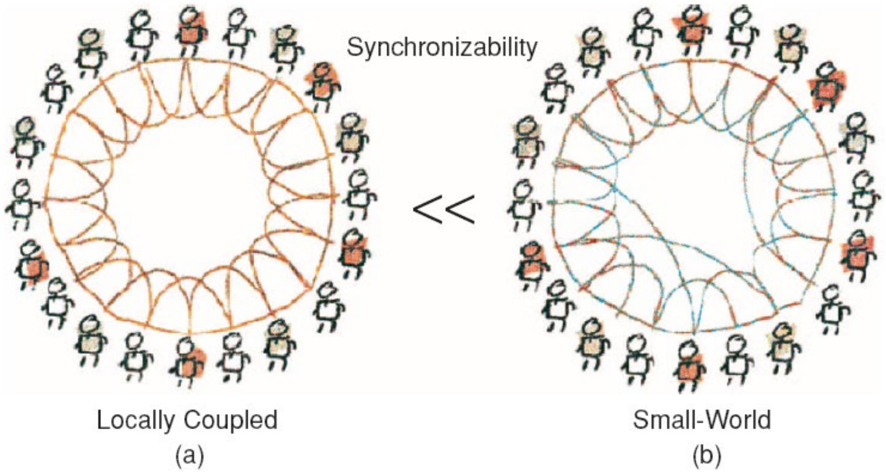


Watts-Strogatz "loop":

$$\lambda_2 \to 0 \text{ per } N \to \infty$$

Synchronization is more difficult as N grows.

In a Watts-Strogatz loop, synchronization can be greatly favoured by adding a few “long distance” connections (small-world network).



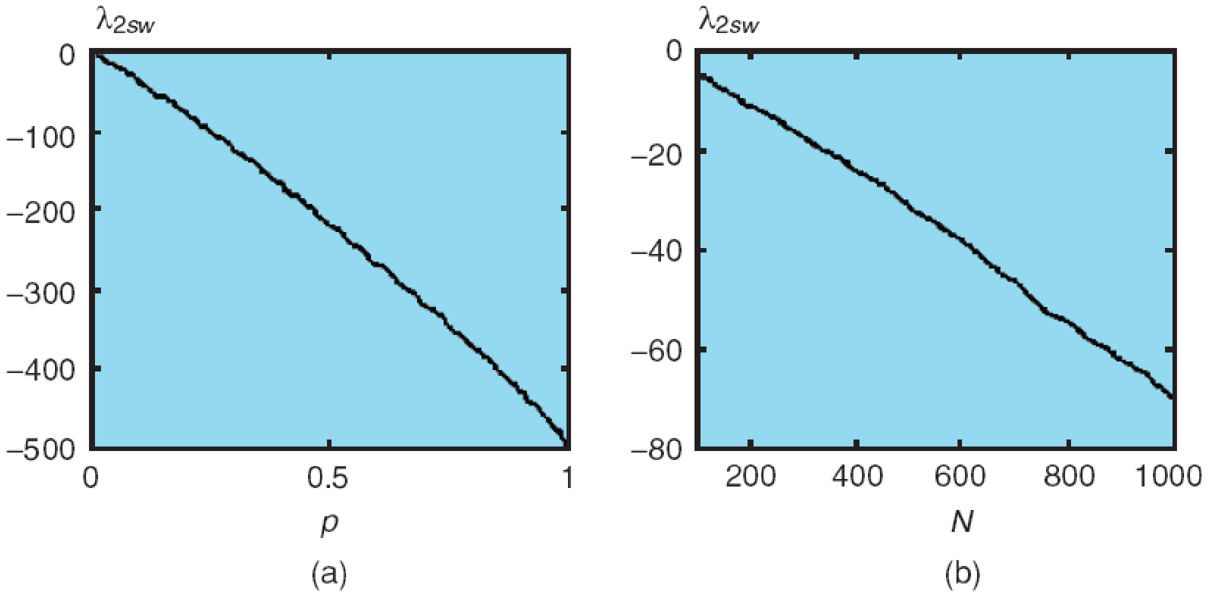
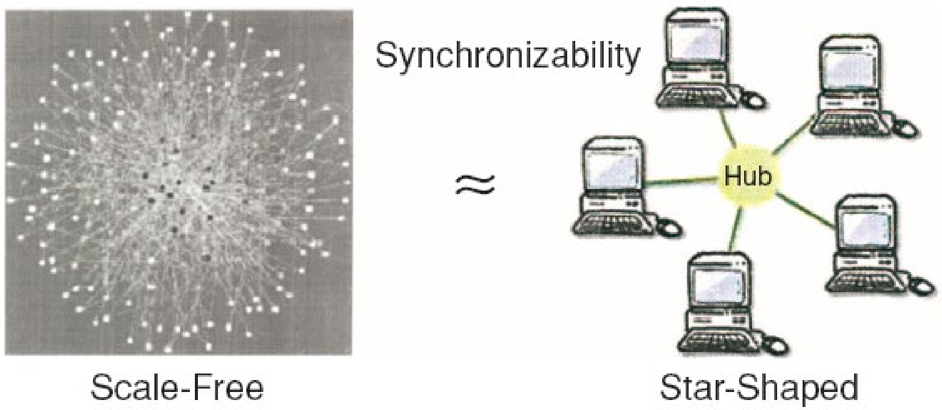


Figure 14. The second-largest eigenvalue λ_{2sw} of the coupling matrix of the small-world network (1) [41]. (a) λ_{2sw} as a function of the adding probability p with the network size $N = 500$. (b) λ_{2sw} as a function of the network size with adding probability $p = 0.1$.

In a scale-free network λ_2 is close to 1, similarly to a pure star network ($\lambda_2 = 1$).



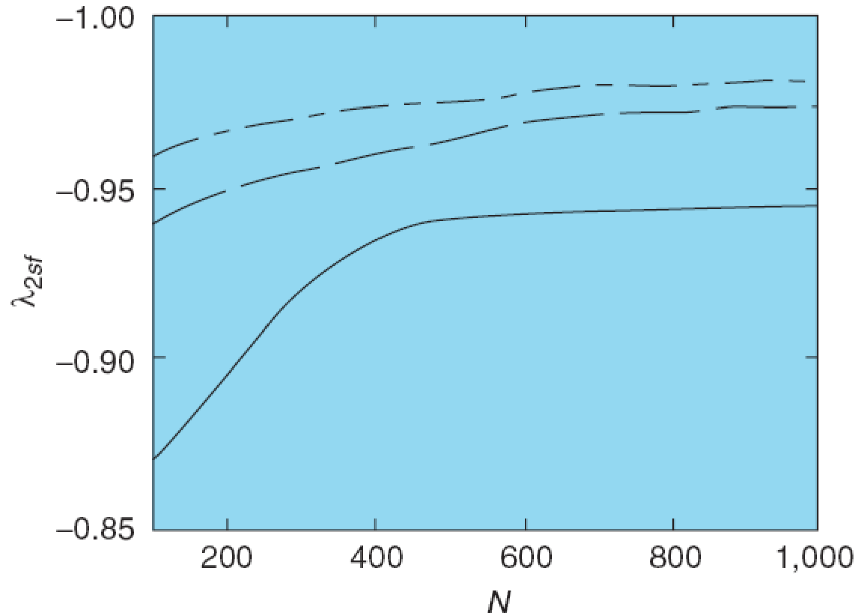
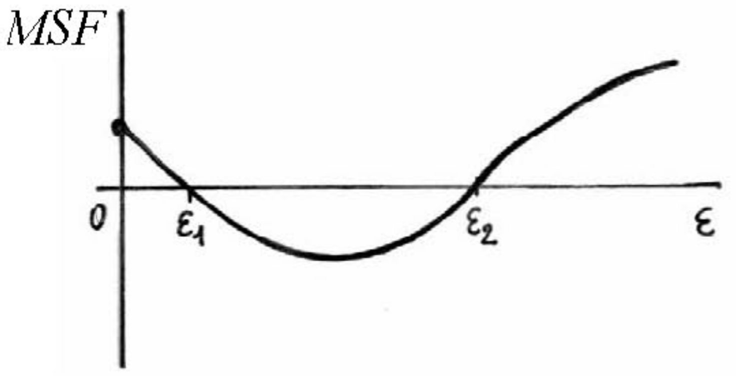


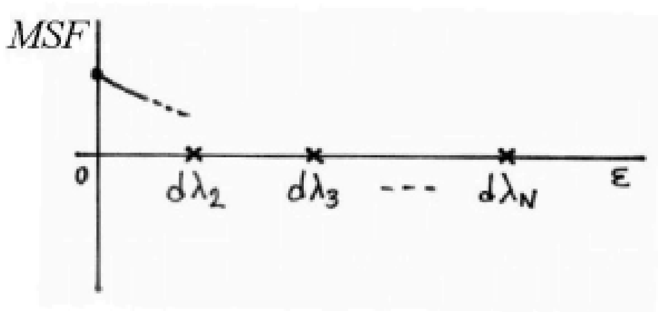
Figure 16. The second-largest eigenvalue of the coupling matrix of the scale-free network (1), for $m_0 = m = 3$ (−); $m_0 = m = 5$(— —); and $m_0 = m = 7$(—·) [42].

MSF type III:



synchronization is possible (with suitable d) if

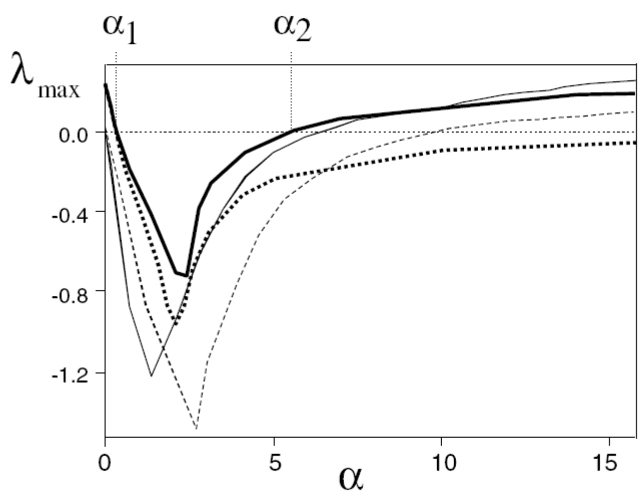
$$\frac{\lambda_N}{\lambda_2} < \frac{\varepsilon_2}{\varepsilon_1}$$



Remark: it is not true that any network can be synchronized provided d is sufficiently large:

- in some networks, when d grows we have first synchronization and then de-synchronization
- some networks cannot be synchronized by any d

FIG. 1. Four typical master stability functions for coupled Rössler oscillators: chaotic (bold) and periodic (regular lines); with y coupling (dashed) and x coupling (solid lines). (All scaled for clearer visualization.) We concentrate on the x-coupled chaotic case with a negative region (α_1, α_2).



Given a MSF type III, synchronization is favoured in networks with small λ_N / λ_2.

In a Watts-Strogatz "loop", λ_N / λ_2 diverges with N ($\lambda_N / \lambda_2 \cong \alpha N^2 /(m(m+1))$).

Again, synchronization can strongly be favoured (at constant N) by adding a few “long distance” connections (small-world network).

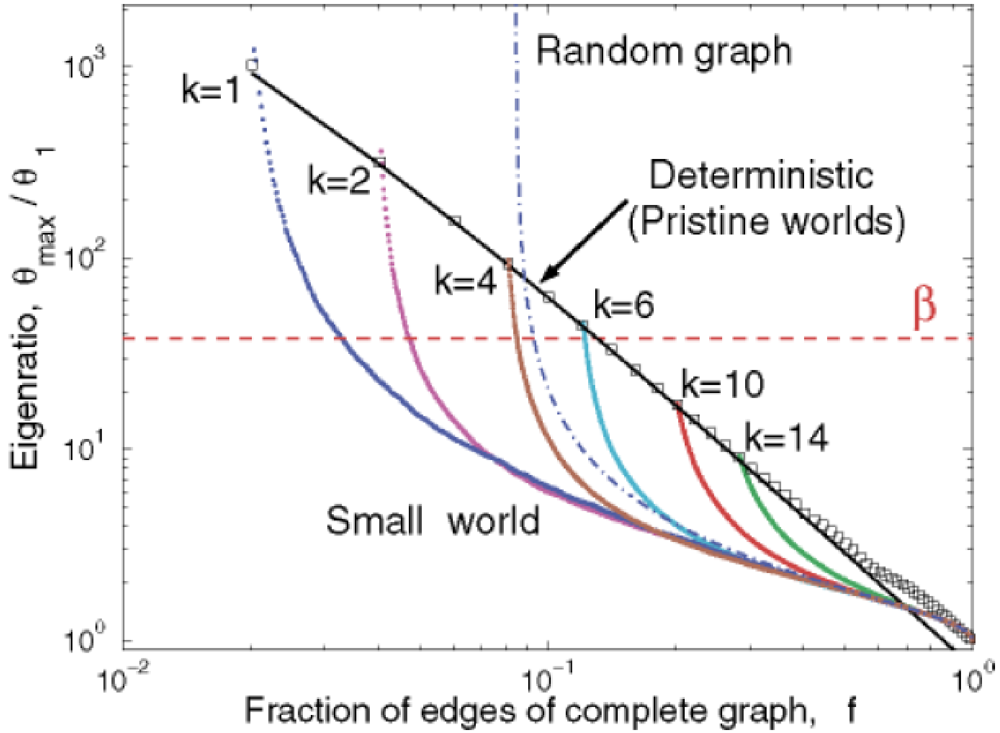
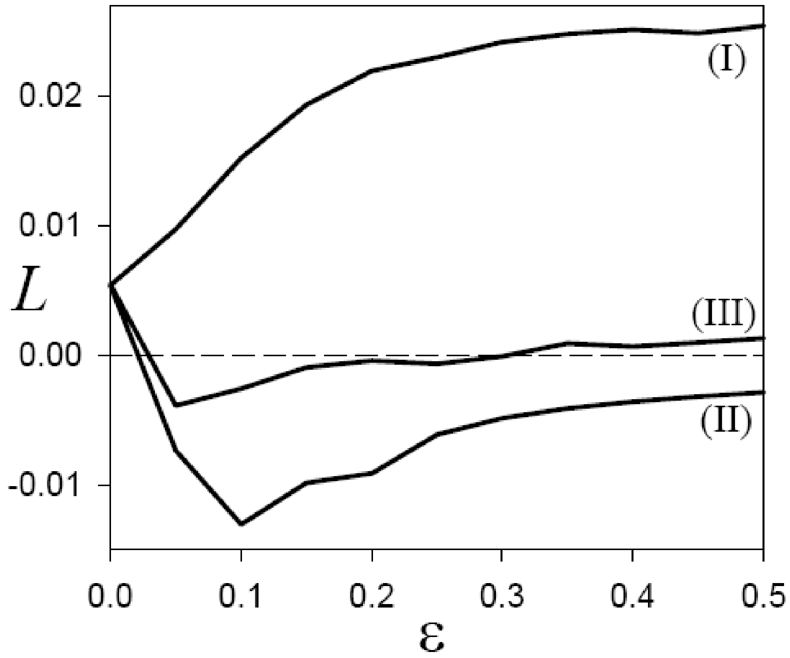


FIG. 2 (color online). Decay of the eigenratio in a $n = 100$ lattice as $f\binom{n}{2}$ edges are added following purely deterministic, semirandom (SW), and purely random schemes. Networks become synchronizable below the dashed line (β). The squares (numerical) and the solid line [analytic Eq. (4)] show the eigenratio decay of pristine worlds through the deterministic addition of short-range connections. The dot-dashed line corresponds to purely random graphs [Eq. (5)], which become *almost surely* disconnected and unsynchronizable at $f \simeq 0.0843$. The semirandom SW approach (dots, shown for ranges $k = 1, 2, 4, 6, 10, 14$) is more efficient in producing synchronization when $k < \ln n$.

Example: three-trophic food chain



- (I) $H = H' = diag[1 \;\; 0 \;\; 0]$: synchronization is impossible for any network
- (II) $H = H'' = diag[0 \;\; 1 \;\; 0]$: synchronization is possible for any network, provided d be sufficiently large
- (III) $H = H''' = diag[0 \;\; 0 \;\; 1]$: synchronization is possible (with suitable d) only for some networks