PHASE SYNCHRONIZATION AND COMPLETE SYNCHRONIZATION

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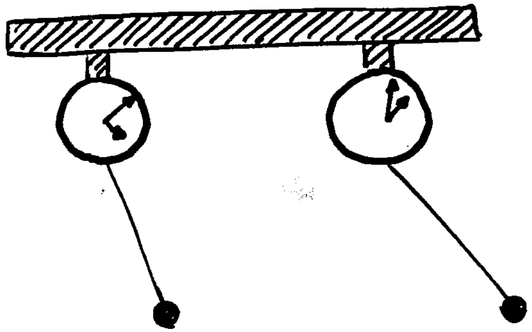
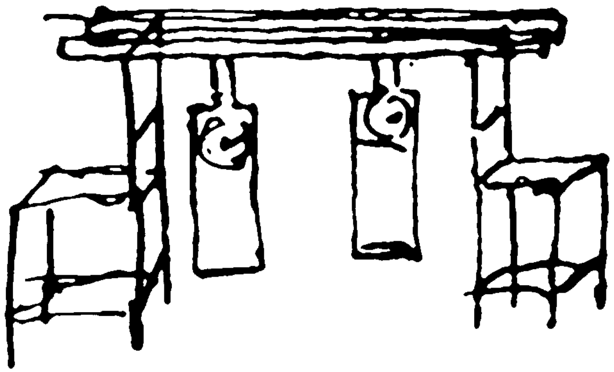
SYNCHRONIZATION

Synchronization is the adjustment of the rhythms of two (or more) oscillators due to their weak interaction.

Example: pendulum clocks (Huygens, 1665)





When independent, the two clocks (=oscillators) have slightly different velocities.

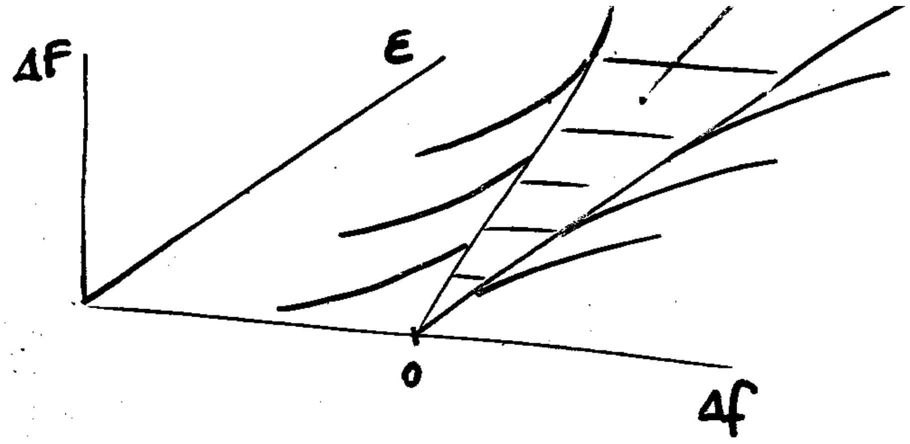
When connected through a common support (=weak interaction), the clocks have perfectly identical velocities (=rhythm adaptation).

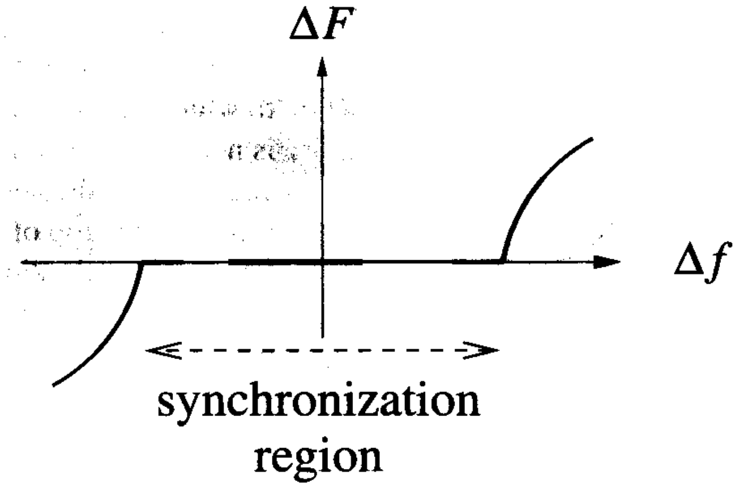
The bifurcation scenario is that of an Arnold tongue:

f_1, f_2: frequencies of oscillators non interacting ($\Delta f = f_1 - f_2$)

F_1, F_2: frequencies of oscillators interacting ($\Delta F = F_1 - F_2$)

ε: coupling strength

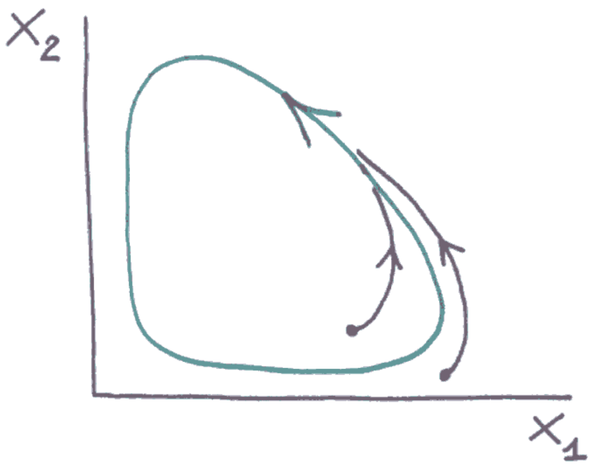


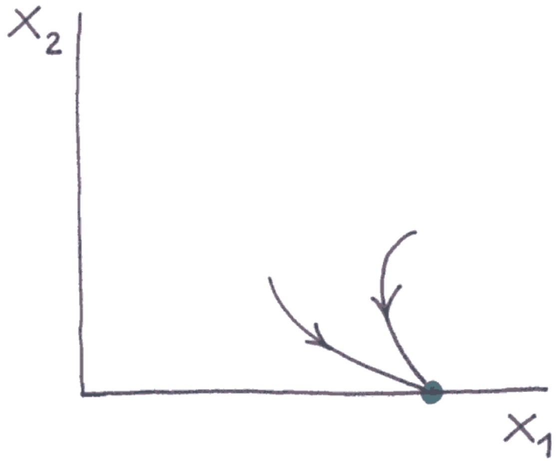


- Synchronization ($\Delta F = 0$) can only take place if $|\Delta f|$ is sufficiently small.
- Synchronization can take place with arbitrary small ε (but needs smaller $|\Delta f|$).

Observing two variables oscillating in a synchronous way does not necessarily imply the existence of two synchronized systems.

Example: prey-predator system



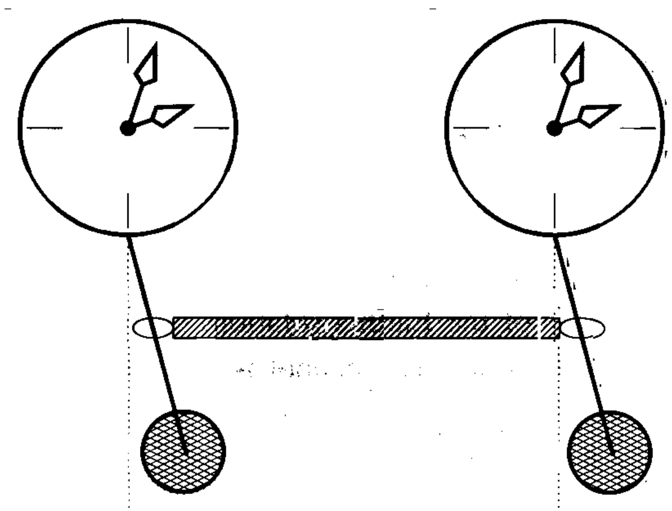


x_1=n° of preys
x_2=n° of predators

The oscillations disappear if the two variables are decoupled.

The system is not separable into (non interacting) subsystems able to oscillate independently.

The synchronous behaviour of two systems cannot properly be qualified as synchronization if the interaction is "strong".



When the interaction can be qualified as weak?

As a guideline, if a subsystem stops oscillating it should not prevent the other one to continue do it.

PHASE SYNCHRONIZATION OF PERIODIC OSCILLATORS

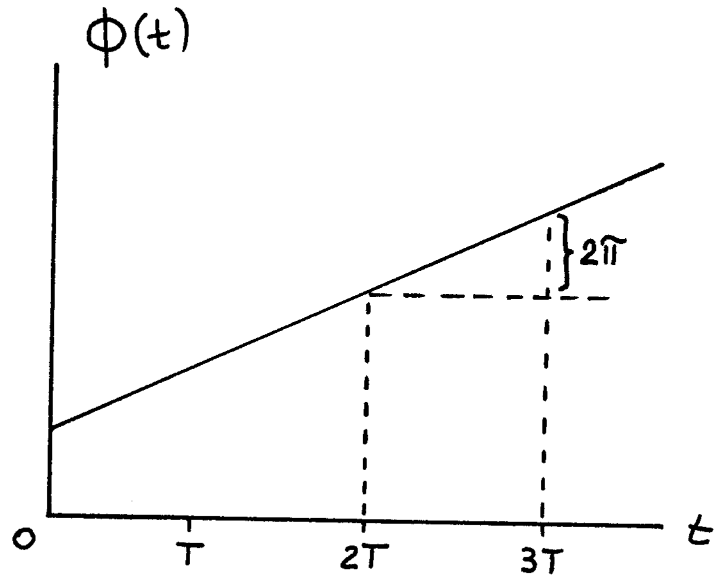
Phase of periodic oscillators

The phase $\Phi(t)$ of an oscillator with period T is defined as:

$$\Phi(t) = \frac{2\pi t}{T} + \text{const} = \omega t + \text{const}$$

$\Phi(t)$ grows linearly in time.

$\omega = \mathrm{d}\Phi(t)/\mathrm{d}t$ is the (angular) frequency of oscillation.



Synchronization of a periodic oscillator with a periodic forcing input

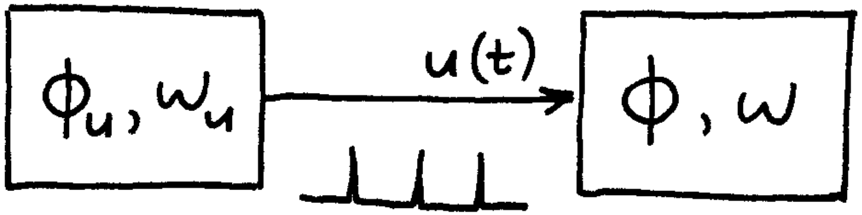
A periodic oscillator subject to a periodic forcing input $u(t)$:

$\Phi_u(t)$=phase of the forcing input

$\omega_u = \mathrm{d}\Phi_u(t)/\mathrm{d}t$

$\Phi(t)$=phase of the oscillator

$\omega = \mathrm{d}\Phi(t)/\mathrm{d}t$



We assume that $\omega \neq \omega_u$ when there is no interaction ($u(t)=0$).

The oscillator is synchronized with the forcing input (phase synchronization, or phase locking) when

$$\Phi(t) - \Phi_u(t) = \text{const}$$

Remark: $\Phi(t)-\Phi_u(t)=\text{const}$ if and only if

$$\omega=\omega_u$$

Thus phase synchronization is equivalent to frequency synchronization ("frequency locking").

In the synchronized regime, the oscillator is "locked" to the frequency of the forcing input. The amplitude of the two signals may remain uncorrelated.

More in general, we have frequency synchronization of order $n:m$ (n,m integers) when

$$n\omega_u=m\omega \qquad \text{that is} \qquad n\Phi_u(t)-m\Phi(t)=\text{const}$$

The oscillators counts n periods for each m periods of the forcing input ($nT=mT_u$).

Example: forced respiration by mechanical ventilation

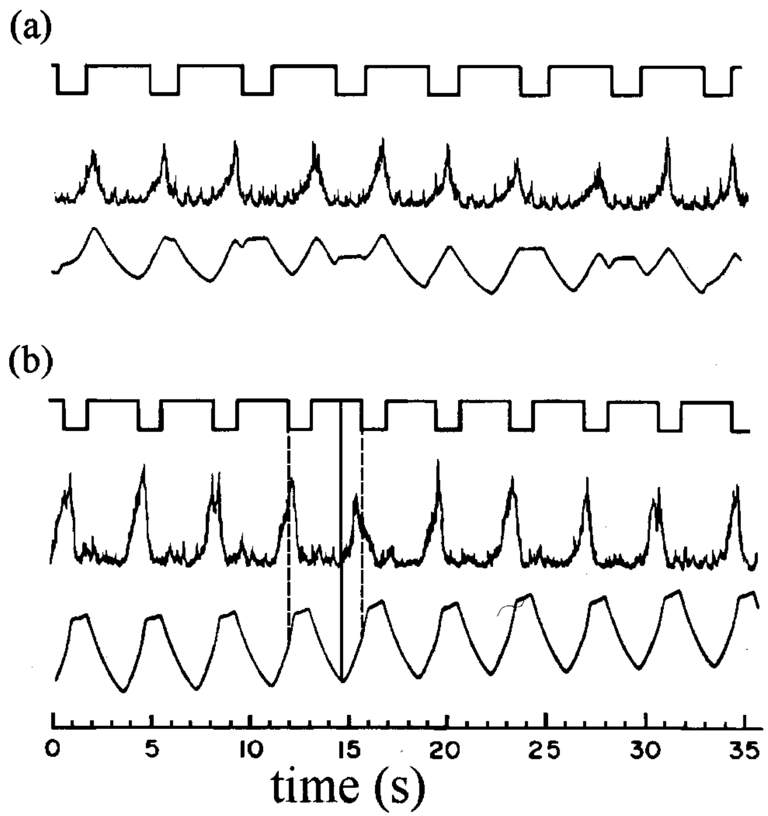
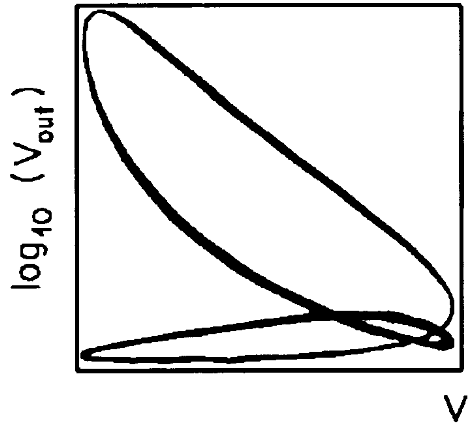


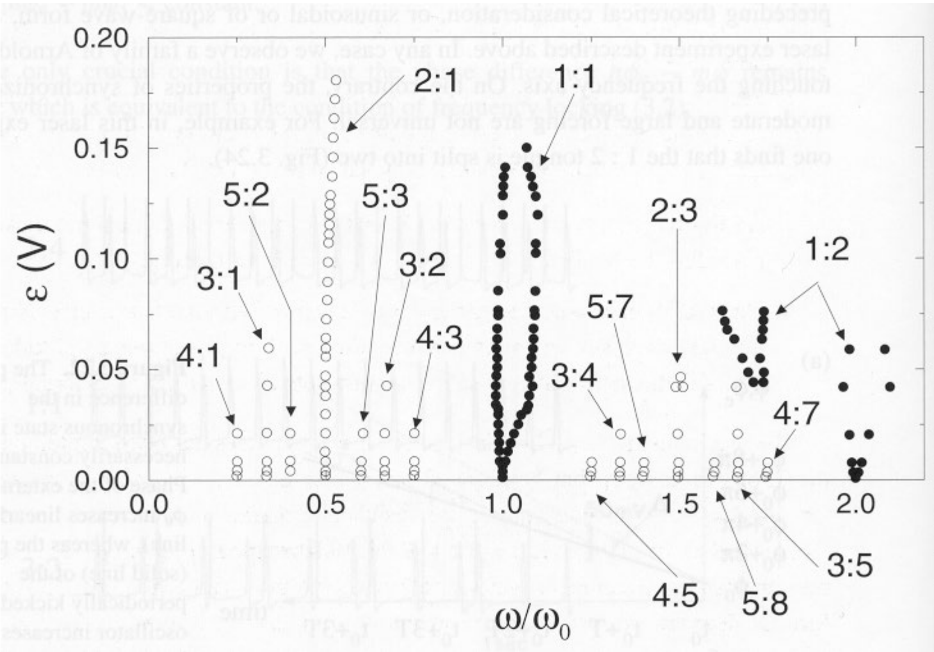
Figure 3.10. Nonsynchronous (a) and synchronous (b) breathing patterns. Top curves show mechanical ventilation (downwards corresponds to inflation). Middle curves are the preprocessed electromyograms of the diaphragm; these signals reflect the output of the central respiratory activity. Lower curves show the ventilation volume. Dashed and solid lines in (b) show the onset of inflation and inspiration, respectively. Note that entrainment of the spontaneous respiration by the external force (mechanical inflation) result in periodic variation of the ventilation volume (b).

Example: synchronization of various orders $n:m$ in a laser

- Unforced system: light intensity oscillates with $f \cong 40\text{Hz}$.
- Periodic (forcing) input (voltage signal) with various frequencies and amplitudes.

Synchronization 1:2: 1 period of the oscillator every 2 periods of the forcing input.

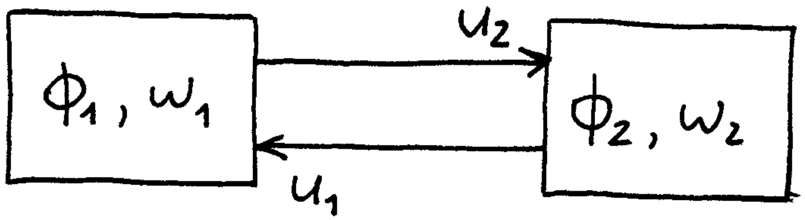




Arnold tongues for synchronizations of various orders.

Synchronization of two periodic oscillators

The influence is bidirectional:



$\Phi_1(t)$ = phase of oscillator 1 ($\omega_1 = \mathrm{d}\Phi_1(t)/\mathrm{d}t$)
$\Phi_2(t)$ = phase of oscillator 2 ($\omega_2 = \mathrm{d}\Phi_2(t)/\mathrm{d}t$)

We assume that $\omega_1 \neq \omega_2$ when there is no interaction ($u_1 = u_2 = 0$).

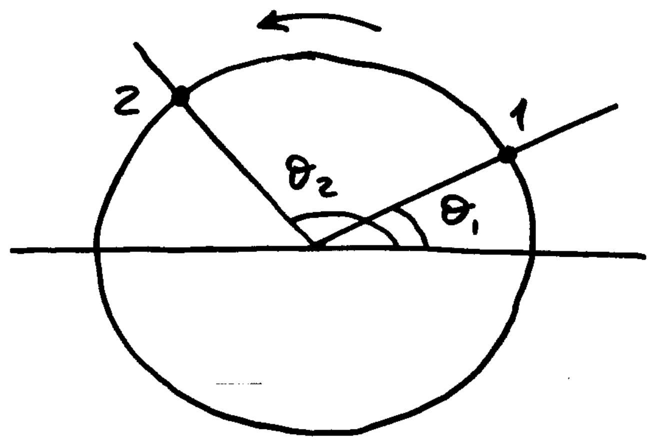
The two oscillators are synchronized (in phase and frequency) when

$$\Phi_1(t) - \Phi_2(t) = \text{const} \qquad \text{that is} \qquad \omega_1 = \omega_2$$

Similarly we define the synchronization of order $n:m$ as

$$n\omega_1 = m\omega_2 \qquad \text{that is} \qquad n\Phi_1(t) - m\Phi_2(t) = \text{const}$$

Example: two athletes are jogging on a circular track, each one with her/his own (constant) speed:



$$\dot{\Phi}_1 = \omega_1$$
$$\dot{\Phi}_2 = \omega_2$$

The system (Φ_1, Φ_2) has (generically) quasi-periodic behaviour.

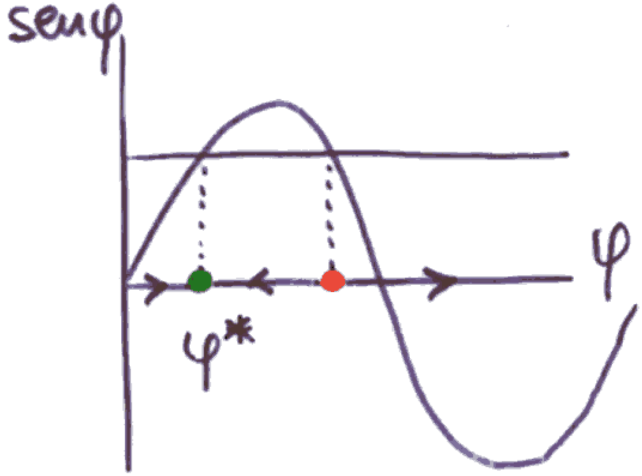
The athletes are friends: they try to synchronize their speeds by correcting their "base" speed with a term dependent on their distance:

$$\dot{\Phi}_1 = \omega_1 + k_1 \sin(\Phi_2 - \Phi_1)$$
$$\dot{\Phi}_2 = \omega_2 + k_2 \sin(\Phi_1 - \Phi_2)$$

The phase difference $\varphi = \Phi_1 - \Phi_2$ evolves according to the 1st-order system:

$$\dot{\varphi} = \omega_1 - \omega_2 - (k_1 + k_2) \sin \varphi$$

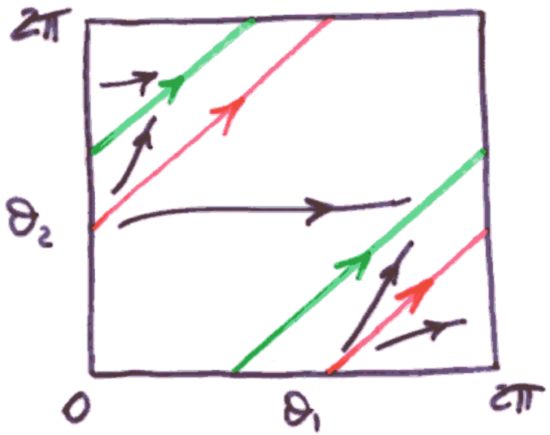
The athletes are phase synchronized when $\dot{\Phi}_1 = \dot{\Phi}_2$, that is



$$\dot{\varphi} = 0 \Leftrightarrow \sin\varphi = \frac{\omega_1 - \omega_2}{k_1 + k_2}$$

Provided $(\omega_1 - \omega_2)$ is sufficiently small and/or $(k_1 + k_2)$ is sufficiently large, there are two equilibria φ, one asymptotically stable (φ^*) and one unstable.

When $\varphi = \varphi^*$, the system (Φ_1, Φ_2) has periodic behaviour and



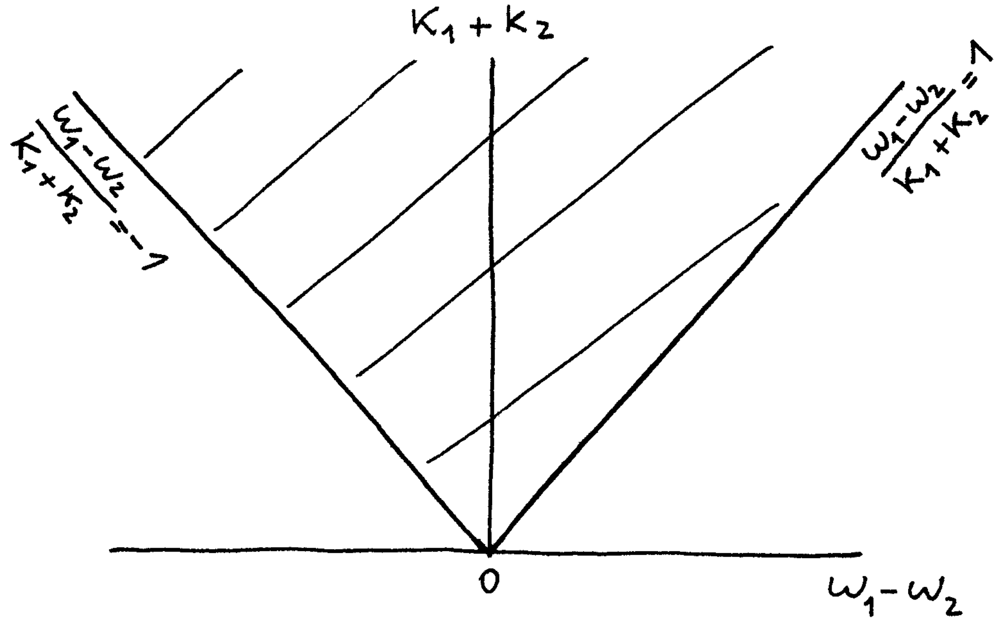
$$\Phi_1(t) - \Phi_2(t) = const$$

But in general $\Phi_1(t) \neq \Phi_2(t)$ (the two athletes run at the same speed but remain distant …).

Phase synchronization (= asymptotically stable periodic solution for (Φ_1, Φ_2)) takes place for

$$\left| \frac{\omega_1 - \omega_2}{k_1 + k_2} \right| < 1$$

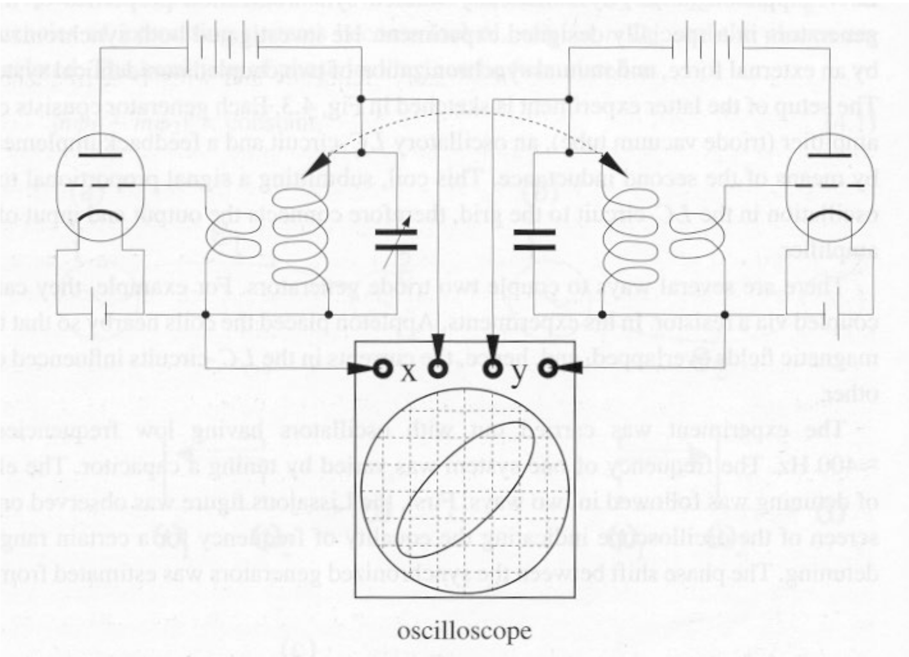
which defines an Arnold tongue.

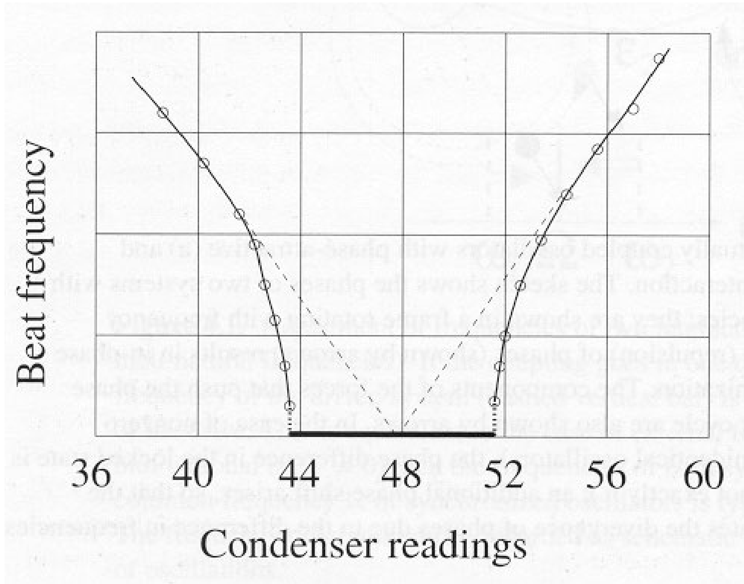


By varying $(\omega_1 - \omega_2)$ and/or $(k_1 + k_2)$, the loss of synchronization takes place - by crossing the border of the Arnold tongue - through a saddle-node bifurcation involving two periodic solutions for (Φ_1, Φ_2), one asymptotically stable and the other unstable.

Example: synchronization of triode oscillators (Appleton, 1922)

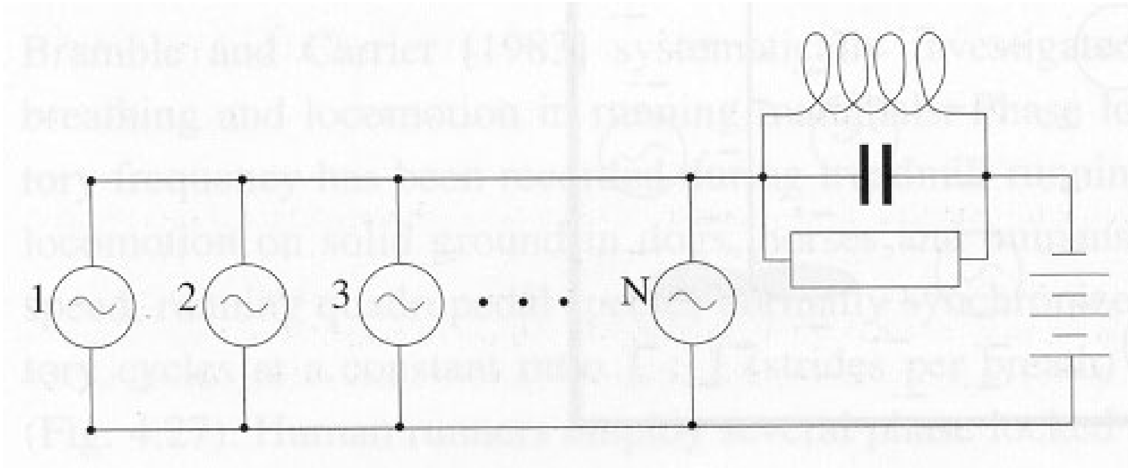
The two oscillators interact through the magnetic fields generated by the inductors (physically close each other).





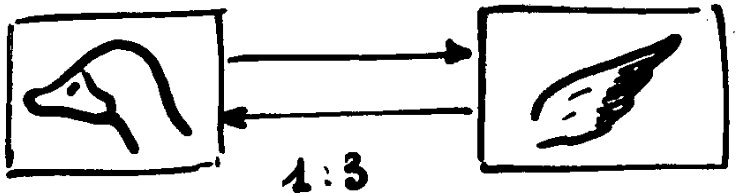
The results of the experiments ($\omega_1 - \omega_2$ as a function of the variable capacity C) show an interval of C where synchronization takes place.

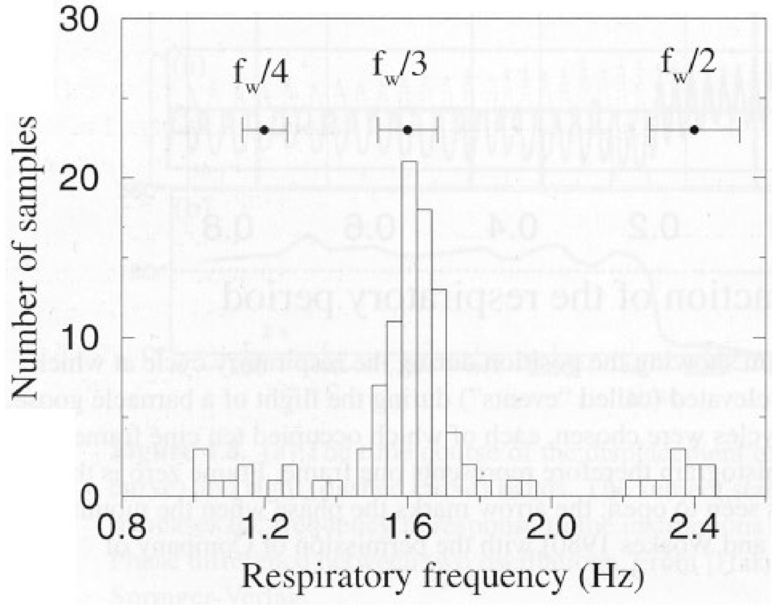
Example: synchronization of electric generators



The generators connected to the power distribution system keep the same speed of rotation (= frequency of the electric signal) thanks to synchronization.

Example: coordination of respiration rhythm and wing beat in the flight of migratory geese

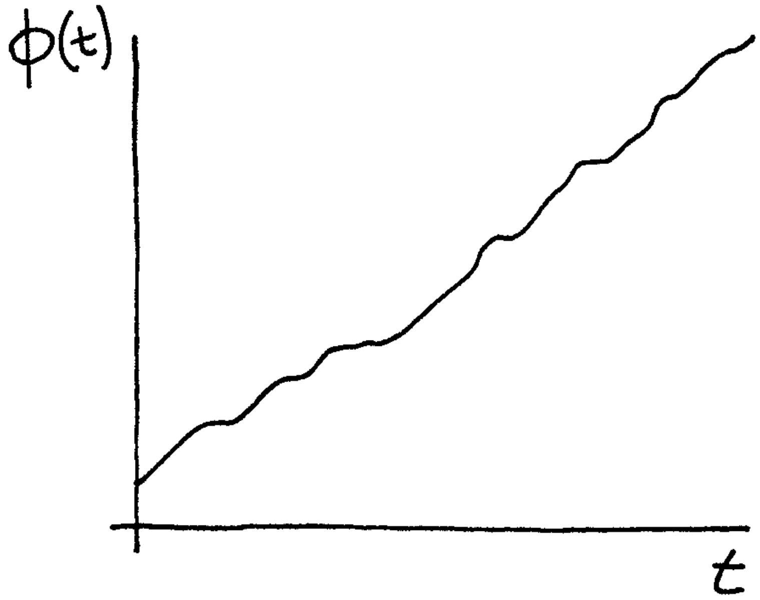




The distribution of the respiratory frequency shows evidence of a synchronization of order 1:3 with the wing beat frequency (1 breath every 3 wing beats).

PHASE SYNCHRONIZATION OF CHAOTIC OSCILLATORS

Phase of chaotic oscillators



The phase $\Phi(t)$ of a chaotic oscillator should be defined as a monotonically increasing variable which parameterizes the system solution.

The average frequency of the oscillator is defined as

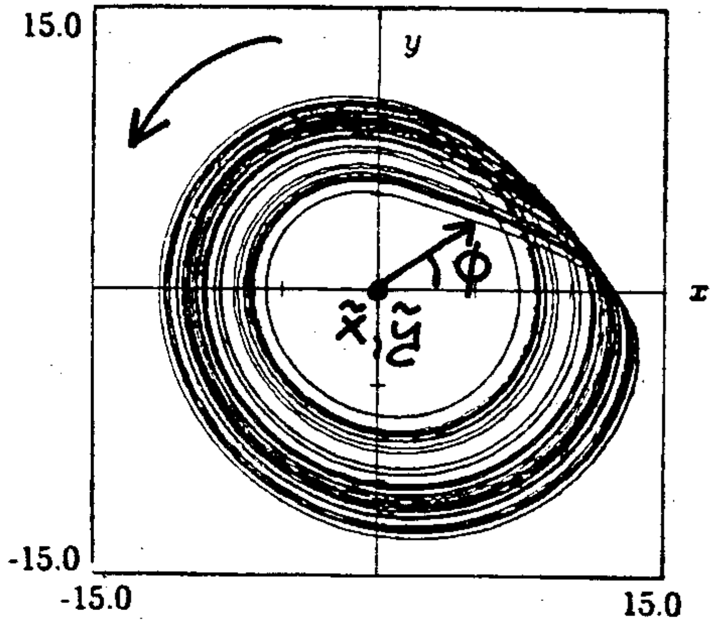
$$\omega = \lim_{t \to \infty} \frac{\Phi(t)}{t}$$

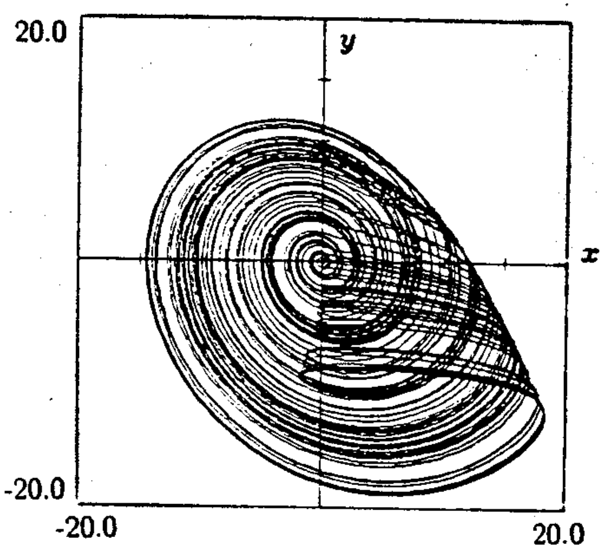
Phase of a chaotic oscillator: projecting the attractor

It is possible when the attractor has a suitable geometric structure.

Example: Rössler system

$$\Phi(t) = \arctan \frac{y(t) - \tilde{y}}{x(t) - \tilde{x}}$$

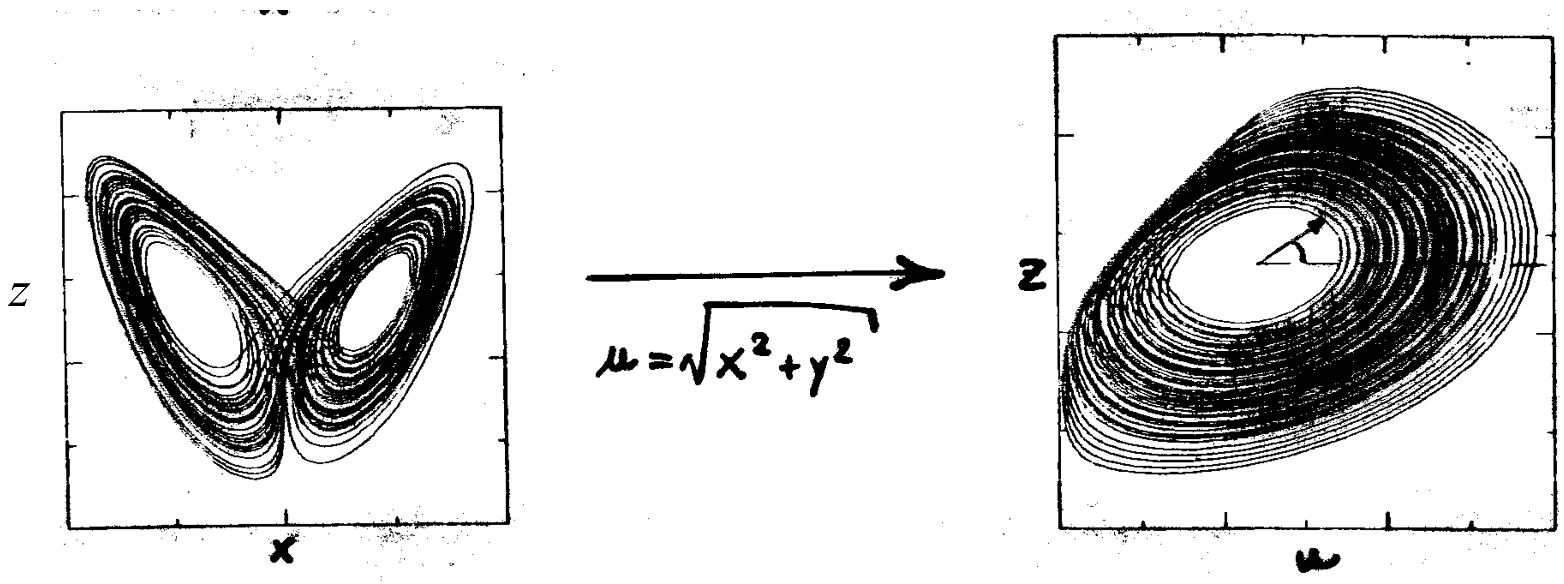




For different parameter values, the above phase definition is impossible due to the shape of the attractor.

In some cases, defining the phase by attractor projection becomes possible after a variable change.

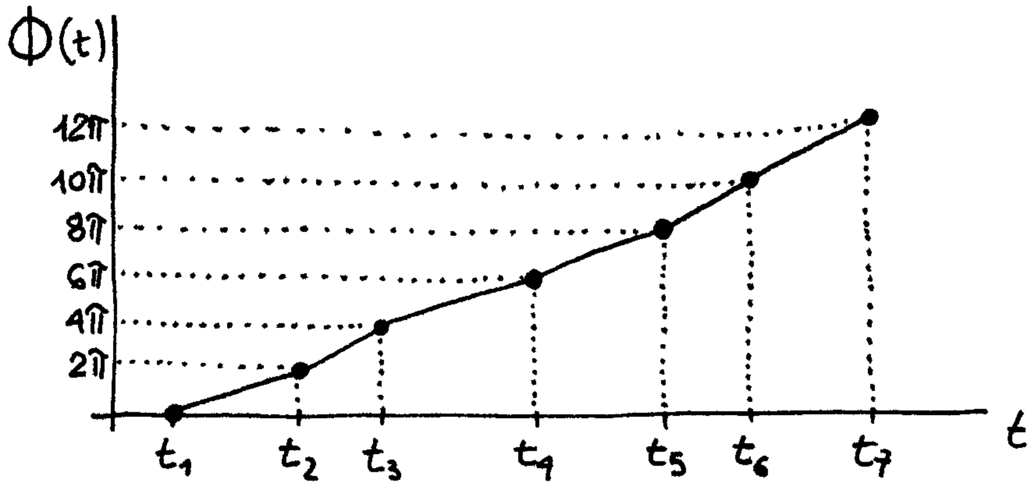
Example: Lorenz system



Phase of a chaotic oscillator: Poincaré section

$\Phi(t)$ is defined to (linearly) increase of 2π between two consecutive crossings of a suitably defined Poincaré section Π.

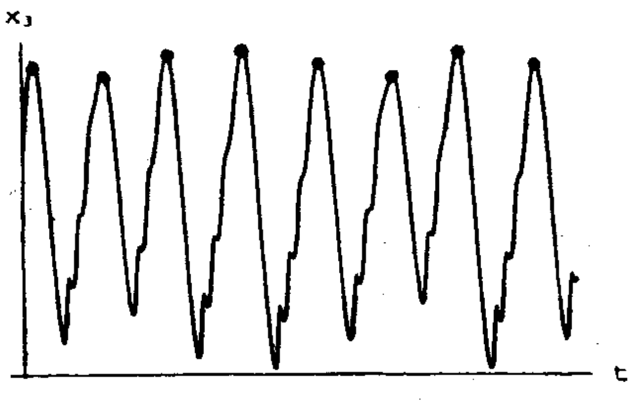
$$\Phi(t) = 2\pi \frac{t - t_k}{t_{k+1} - t_k} + 2\pi k, \qquad t_k \le t < t_{k+1}$$

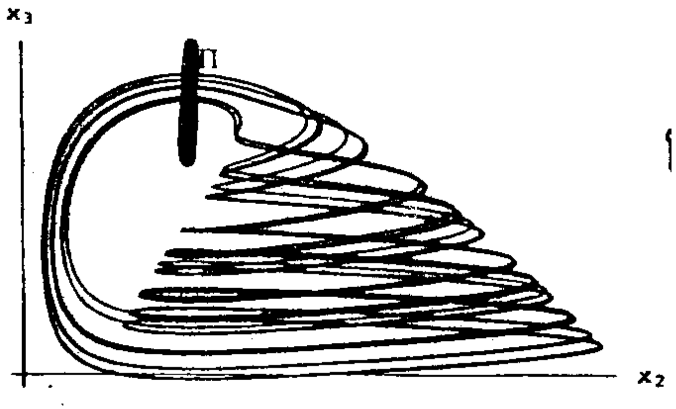


A very practical Poincaré section corresponds to the maxima of one of the state variables.

This allows one to associate a phase variable $\Phi(t)$ to a (chaotic) time series (even if no model is available).

Example: prey-predator system (Rosenzweig-MacArthur)





Synchronization of a chaotic oscillator by a periodic forcing input

- $\Phi_u(t)$ is the phase of the periodic forcing input ($\omega = \mathrm{d}\Phi_u(t)/\mathrm{d}t$)
- $\Phi(t)$ is the phase of the chaotic oscillator ($\Omega = \lim_{t\to\infty} \Phi(t)/t$)

The oscillator is synchronized with the input (phase synchronization) if

$$\left|\Phi(t) - \Phi_u(t)\right| < \mathrm{const}$$

The phase difference can vary in time, but should remain bounded.

Remark: $\left|\Phi(t) - \Phi_u(t)\right| < \mathrm{const}$ if and only if

$$\Omega = \omega$$

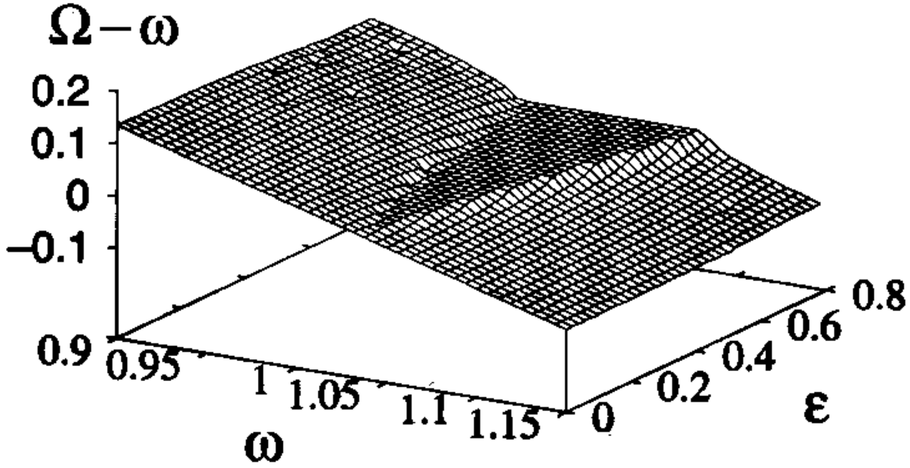
The oscillator is locked to the input frequency - but the amplitudes remain uncorrelated.

The behaviour of the oscillator may remain chaotic.

Example: Rössler system with periodic forcing input

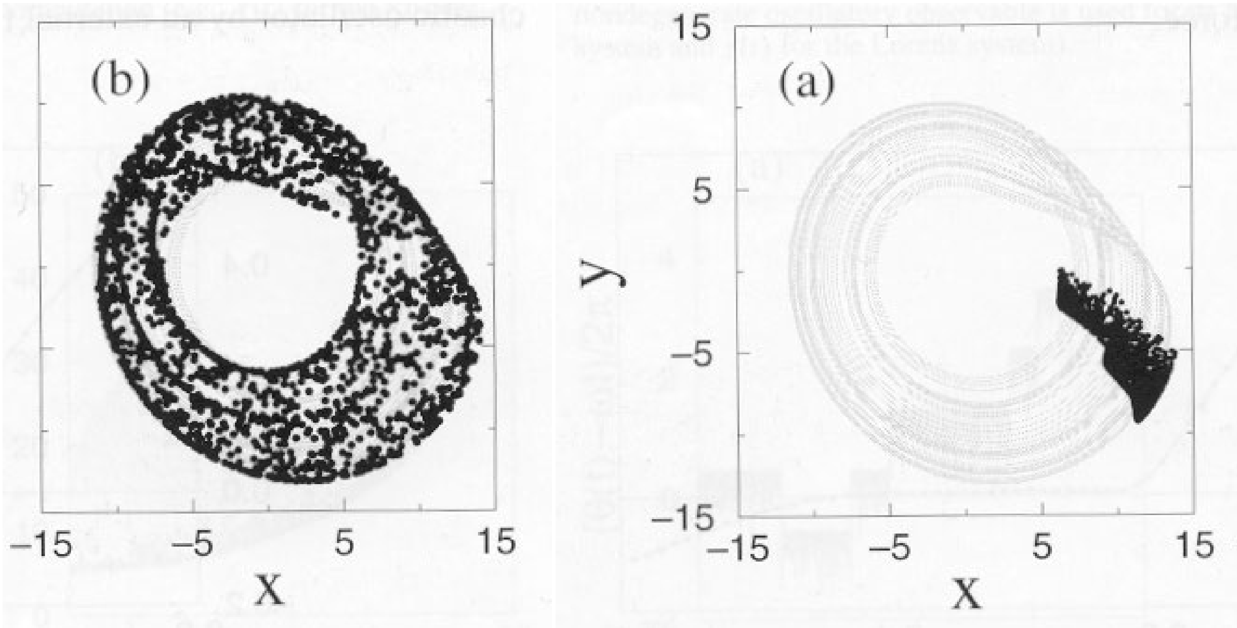
$$\dot{x} = -y - z + \varepsilon \cos(\omega t)$$
$$\dot{y} = x + 0.15 y$$
$$\dot{z} = 0.4 + z(x - 8.5)$$

Phase synchronization takes place in an Arnold tongue.



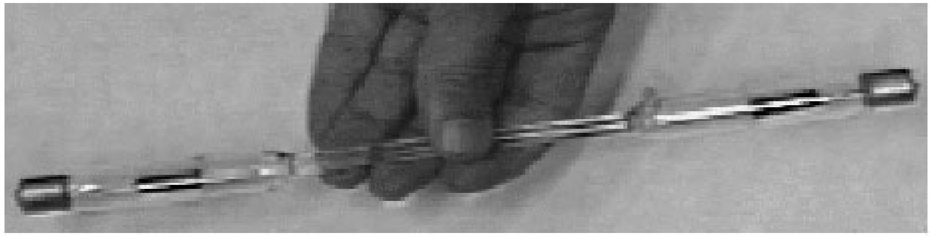
Stroboscopic diagram: dots highlight the state value at time instants multiple of $2\pi/\omega$ (the period of the input).

- Without synchronization (left) dots are spread in all the attractor (their phase is distributed between $-\pi$ and π).
- With synchronization (right) dots are concentrated in a narrow phase interval.

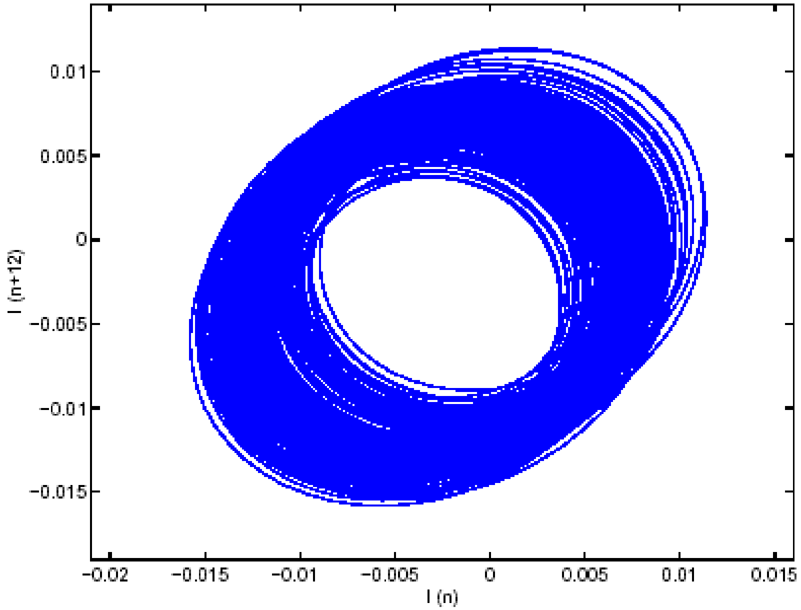


Example: lab experiment: Geissler tube with periodic forcing input

When a constant 800V voltage is applied to the tube (Geissler type with helium), chaotic oscillations in the light intensity are recorded.

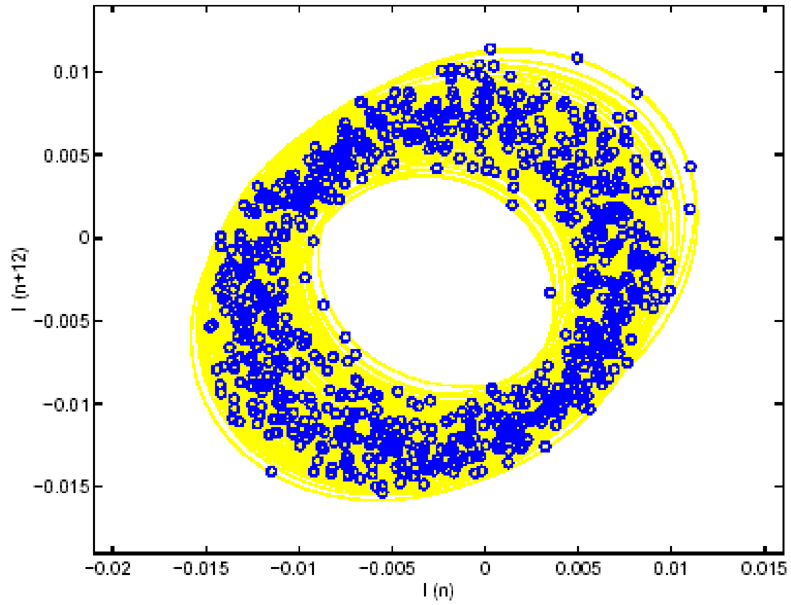


The attractor reconstructed in the two-dimensional space.
The estimate of the fractal dimension is $d = 2.18$.

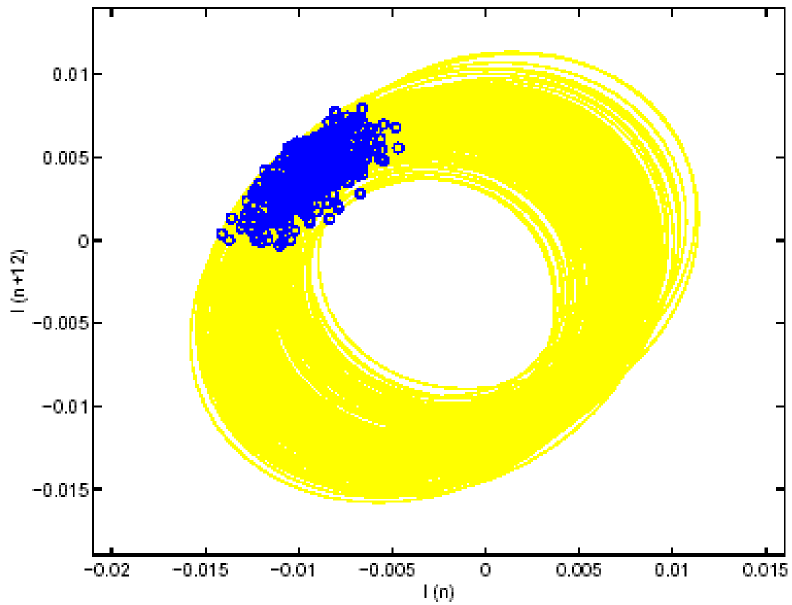


Stroboscopic diagram of the system without input:

dots are spread in all the attractor (their phase is distributed between $-\pi$ and π).



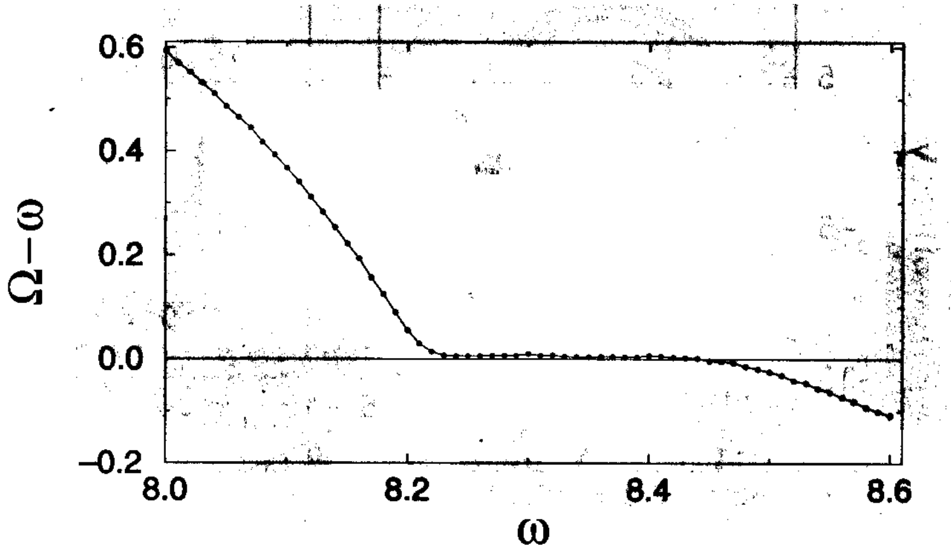
If a small sinusoidal input is applied (amplitude 0.4V, frequency 3850Hz), phase synchronization is obtained.



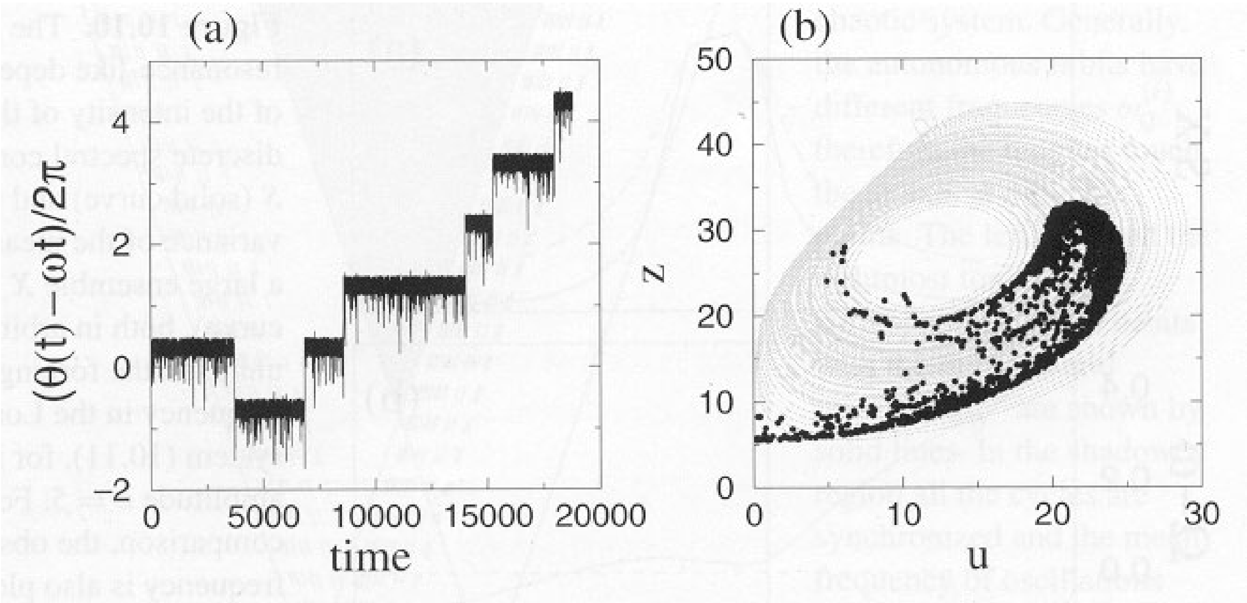
Example: Lorenz system with periodic forcing input

$$\dot{x} = 10(y - x)$$
$$\dot{y} = 28x - y - xz$$
$$\dot{z} = -(8/3)z + xy + \varepsilon\cos(\omega t)$$

Frequency synchronization is "imperfect": $\Omega - \omega \neq 0$ for all ω, ε, but the difference $\Omega - \omega$ is very close to zero for some ω, ε.



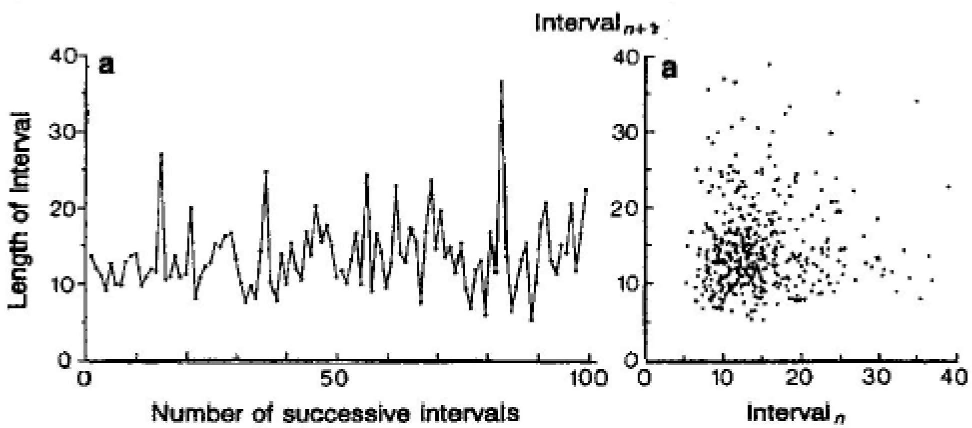
We observe long time intervals of “apparent” synchronization ($|\Phi(t)-\Phi_u(t)| < \text{const}$), interrupted by sudden “jumps” of 2π in $|\Phi(t)-\Phi_u(t)|$ (i.e., sometimes the oscillator "loses un turn" with respect to the input).



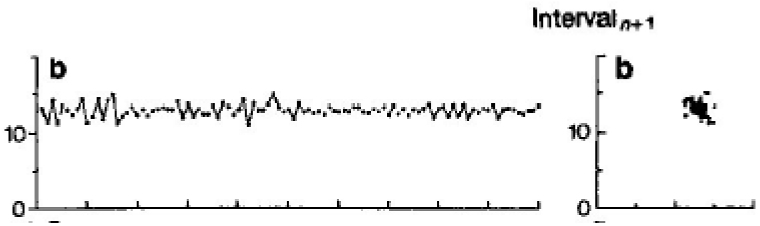
In Lorenz system, this happens when the trajectory comes very close to the saddle $(x,y,z)=(0,0,0)$, where it can be trapped for arbitrarily long time (“saddle effect”).

Example: imperfect synchronization in the locomotion of *Halobacterium salinarium*

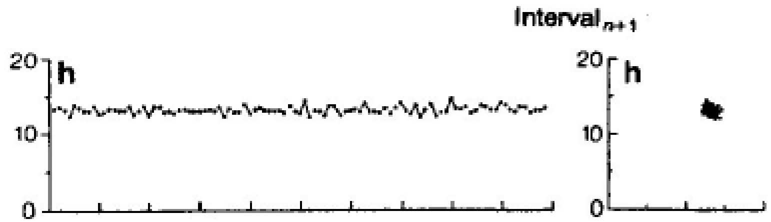
It is a ciliated bacterium moving in a fluid, which commutes direction every few seconds.



Under periodic light stimuli (=flash sequences) of amplitude A, the commuting intervals tend to synchronize with the period T of the stimuli.

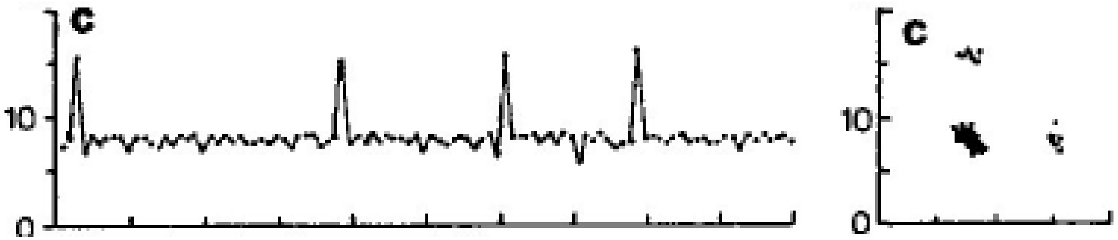


$A = 1 \quad T = 12.5$

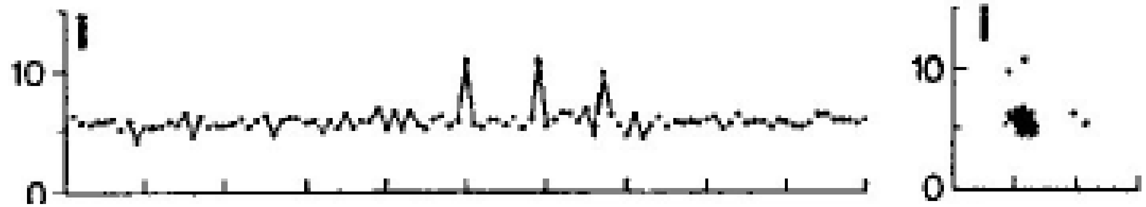


$A = 4 \quad T = 12.5$

However, for some (A,T), some flashes fail to induce commutation.



$A=1 \quad T=9$



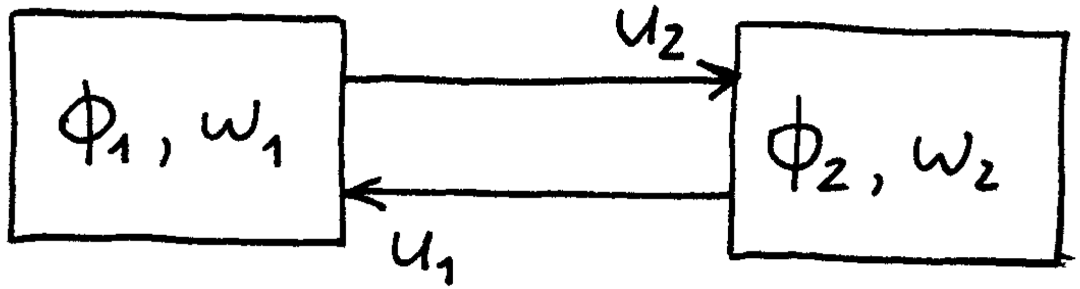
$A=1 \quad T=6$

Imperfect synchronization: long intervals of “apparent” synchronization (=the bacterium commutes with the frequency of the input) are interrupted by sudden “phase jumps” (=the bacterium commutes only once for two flashes).

Synchronization of two chaotic oscillators

$\Phi_1(t)$ = phase of oscillator 1
$(\omega_1 = \lim_{t\to\infty} \Phi_1(t)/t)$

$\Phi_2(t)$ = phase of oscillator 2
$(\omega_2 = \lim_{t\to\infty} \Phi_2(t)/t)$



We assume that $\omega_1 \neq \omega_2$ when there is no interaction ($u_1 = u_2 = 0$).

The two chaotic oscillators are synchronized (in phase and frequency) when

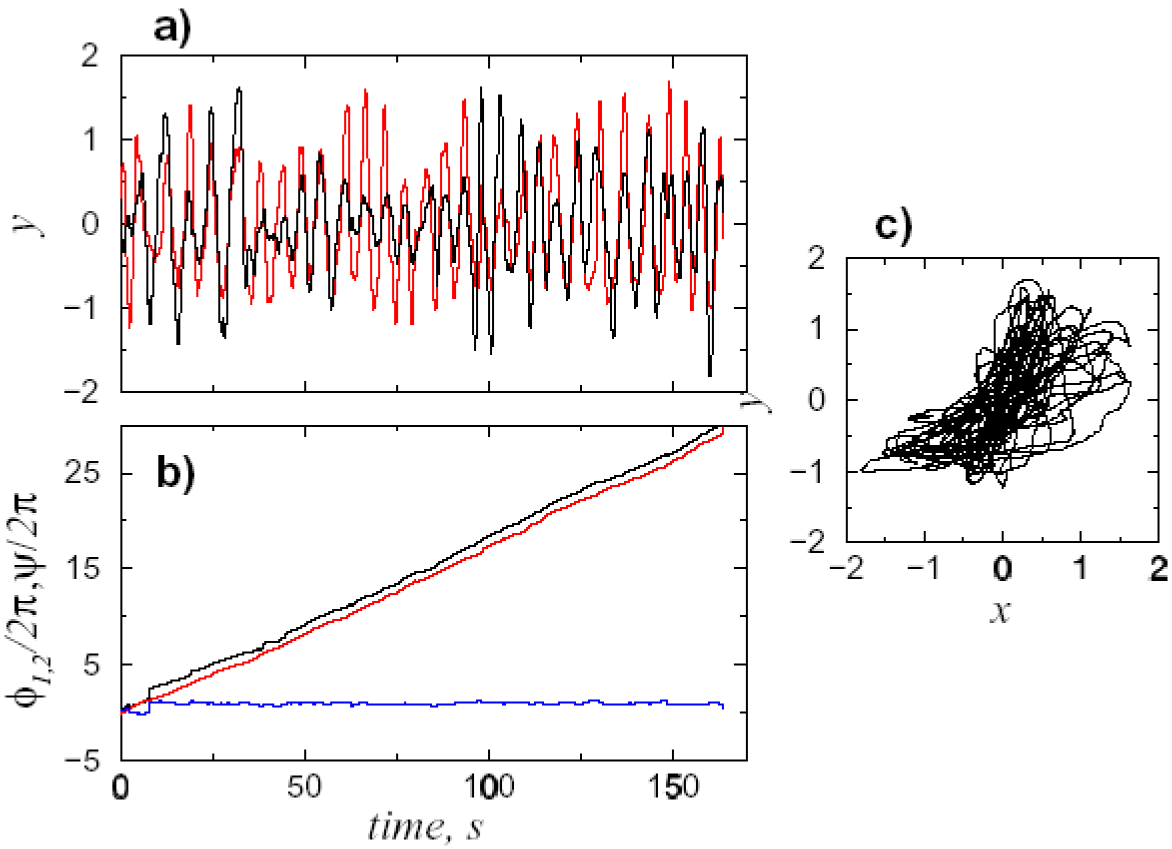
$$|\Phi_1(t) - \Phi_2(t)| < \text{const} \qquad \text{that is} \qquad \omega_1 = \omega_2$$

Example: posture control in humans

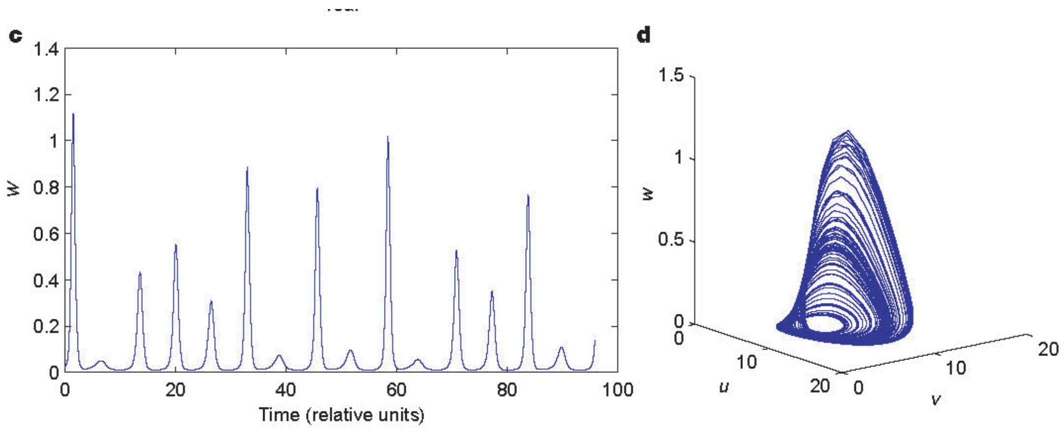
An equipped platform detects the anterior/posterior $x(t)$ and lateral $y(t)$ oscillations of a standing subject, under various experimental conditions (open eyes, closed eyes, etc.).

The bivariate time series $(x(t), y(t))$ ("stabilogram") contain important information on the central nervous system. Typically, correlations among $x(t)$ and $y(t)$ denote the existence of pathologies.

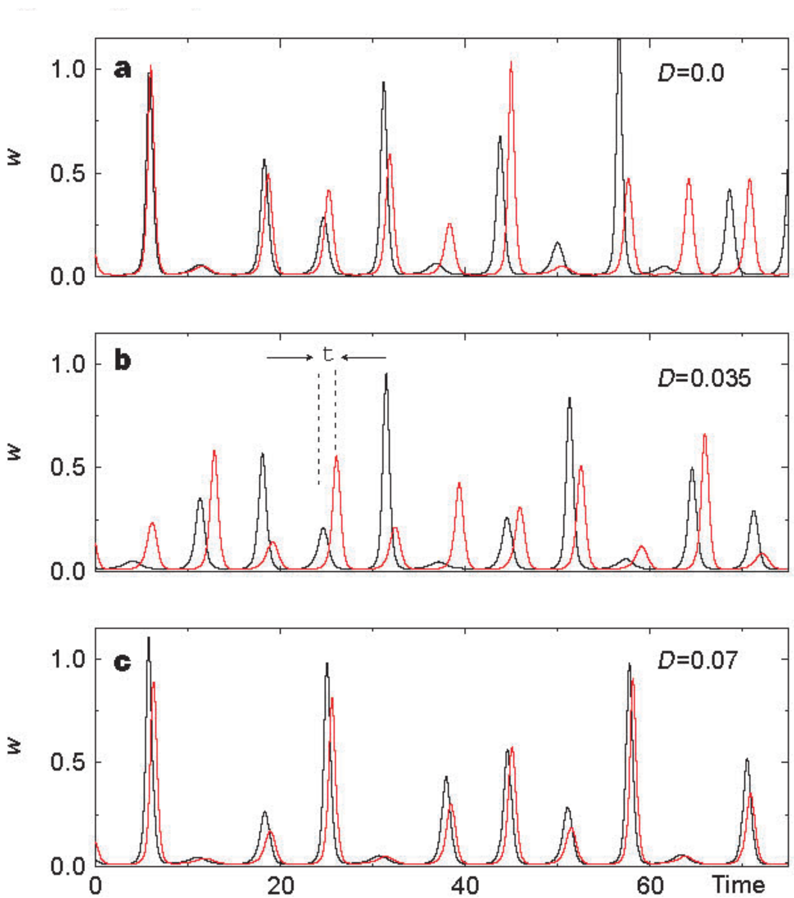
In this example, although the amplitudes of $x(t)$ and $y(t)$ vary in time and appear to be uncorrelated, the two oscillations are perfectly phase synchronized.



Example: synchronization of two food-chain systems (plants/herbivores/predators)



The isolated tri-trophic food chain has chaotic behaviour.



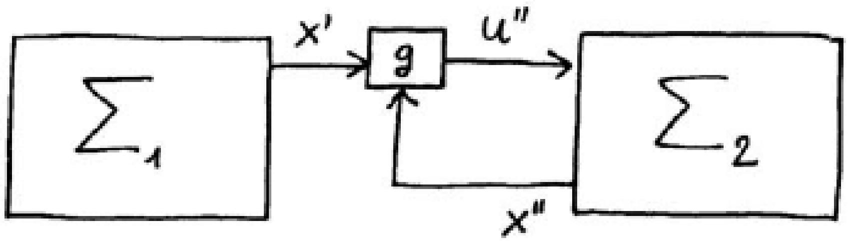
As the coupling D (diffusive migration of herbivores and predators) increases, we observe:

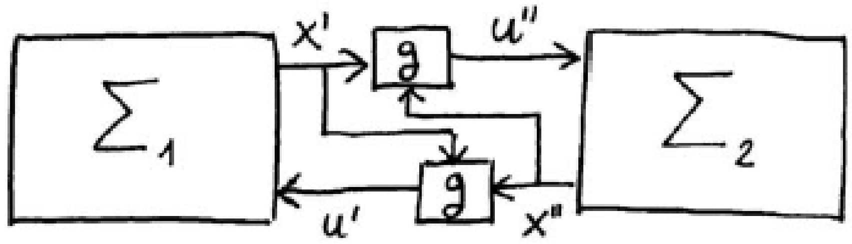
- no synchronization
- phase synchronization (same average frequency, but uncorrelated amplitudes)
- complete synchronization (same average frequency, same amplitudes)

COMPLETE SYNCHRONIZATION

Consider two systems Σ_1 e Σ_2:

- identical (same f): $\dot{x}' = f(x', u')$, $\dot{x}'' = f(x'', u'')$
- in chaotic regime when isolated ($u'(t) = u''(t) = 0 \ \forall t$)
- interacting uni- or bi-directionally: $u' = g(x', x'')$, $u'' = g(x'', x')$





Σ_1 and Σ_2 are completely synchronized if

$$\lim_{t\to\infty} \left| x'(t) - x''(t) \right| = 0$$

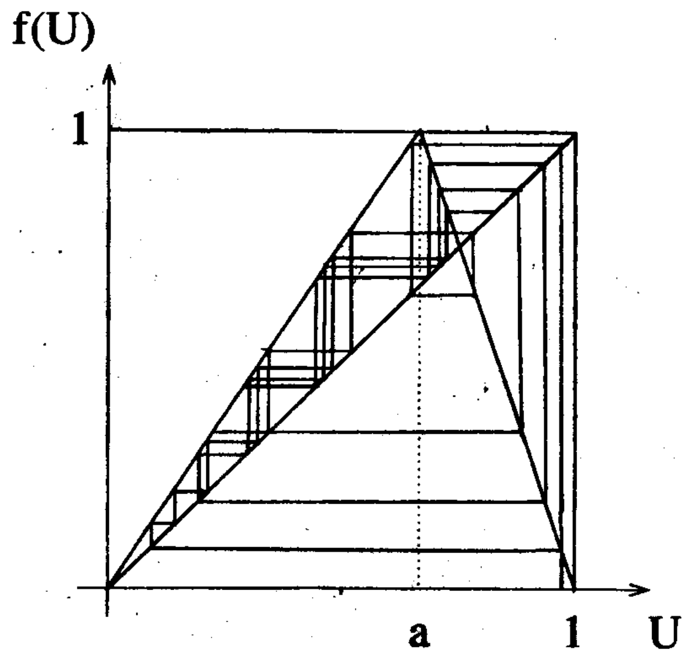
Remarks:

- The definition implies that the "synchronized state" $x'(t) = x''(t)$ be asymptotically stable (at least locally).
- Differently from phase synchronization (=same average frequency but amplitudes not necessarily correlated), complete synchronization implies the perfect coincidence of the behaviours of the two systems.
- For obtaining complete synchronization, the interaction might be non "weak".
- Complete synchronization preserves the chaotic behaviour.
- More in general, the two systems could be non identical but "similar" (e.g., same state equations but slightly different parameters). If so, the requirement is relaxed to $|x'(t) - x''(t)| < \text{constant}$.

Example: synchronization of skewed tent maps

Consider two 1st-order discrete-time systems ("skewed" tent maps, $a = 0.7$):

$$f(U) = \begin{cases} U/a & \text{se } 0 \le U < a \\ (1-U)/(1-a) & \text{se } a \le U \le 1 \end{cases}$$



Both systems (isolated) have chaotic dynamics.

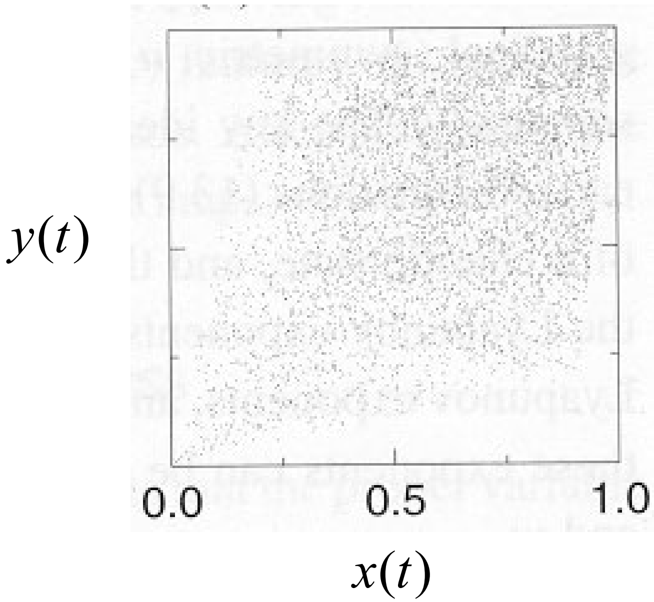
The two systems are coupled bi-directionally:

$$x(t+1) = (1-\varepsilon) f(x(t)) + \varepsilon f(y(t))$$
$$y(t+1) = \varepsilon f(x(t)) + (1-\varepsilon) f(y(t))$$

The parameter ε is the coupling strength:

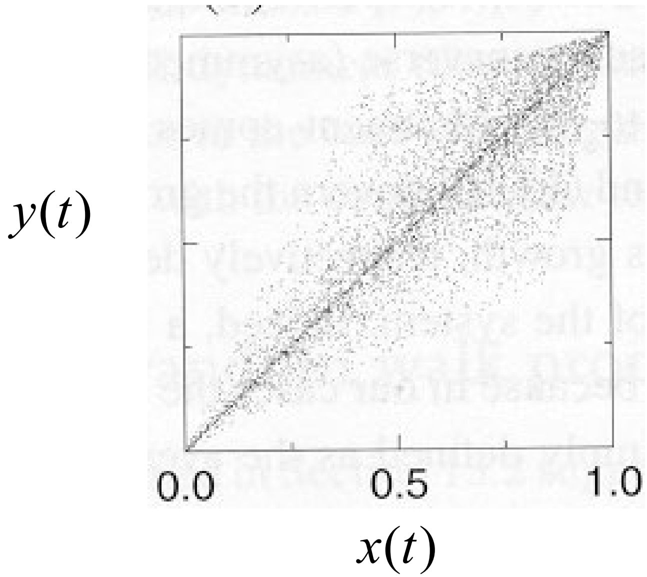
$\varepsilon = 0$ (no interaction)

$x(t)$ and $y(t)$ evolve independently in chaotic regime.



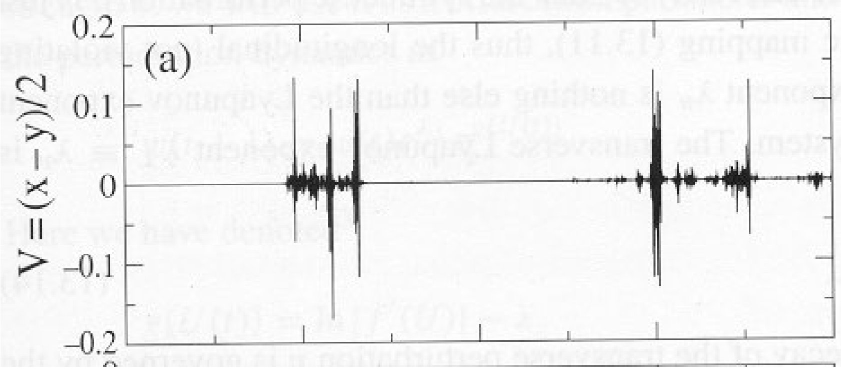
$\varepsilon = 0.2$ (weak interaction)

$x(t)$ and $y(t)$ show the tendency to synchronize.



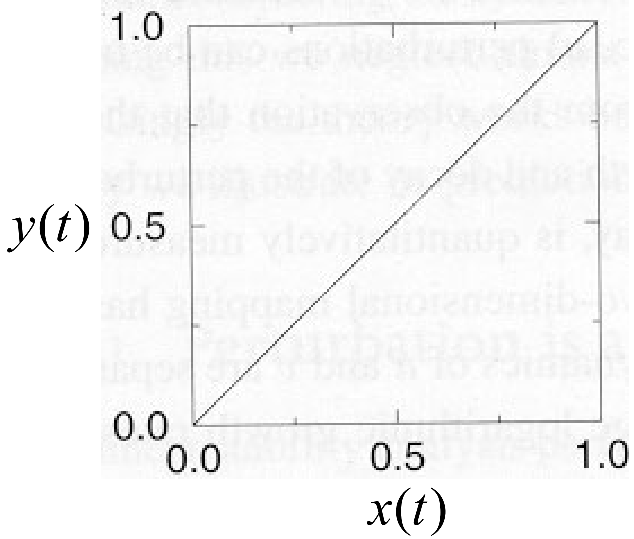
By numerical analysis: complete synchronization takes place for $\varepsilon > \varepsilon_c \cong 0.228$.

With ε slightly smaller than ε_c (e.g. $\varepsilon = \varepsilon_c - 0.001$) we detect intervals of apparent synchronization, interrupted by burst of de-synchronization ("intermittencies").



$\varepsilon = 0.3$ (complete synchronization)

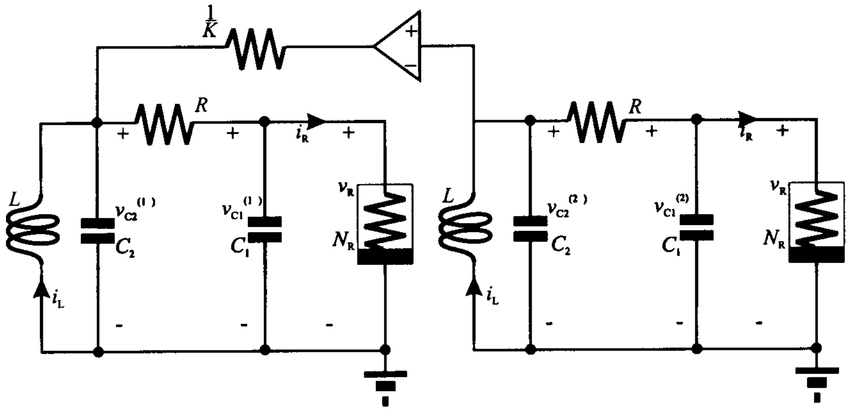
$x(t)$ and $y(t)$ coincide in all time instants and fill the interval (0,1).



$\varepsilon = 1/2$ gives the maximal interaction: we have $x(t) = y(t)$ from $t = 1$ (check the equations!) for all initial conditions (complete synchronization in finite time).

Example: lab synchronization of two Chua circuits

Two circuits identical in theory (slightly different in practice, due to tolerances of the components) interact trough unidirectional coupling:

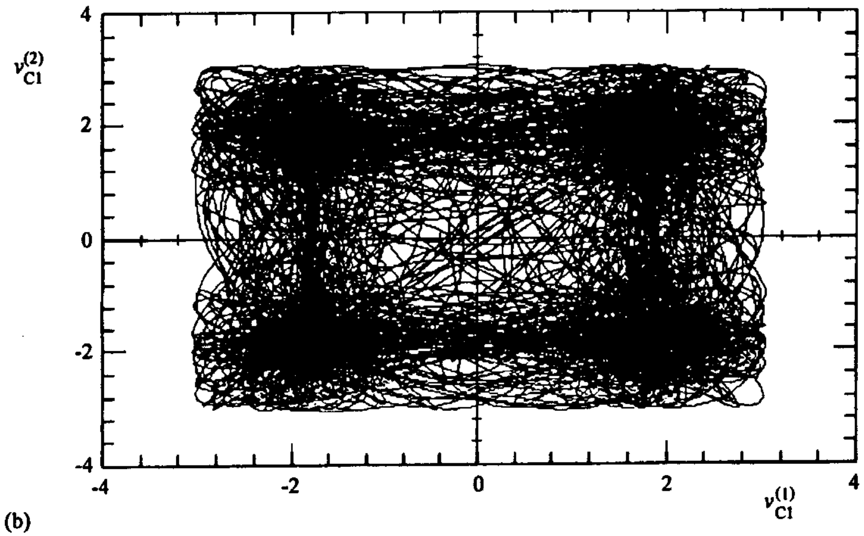


The (adimensional) equations of the two systems are:

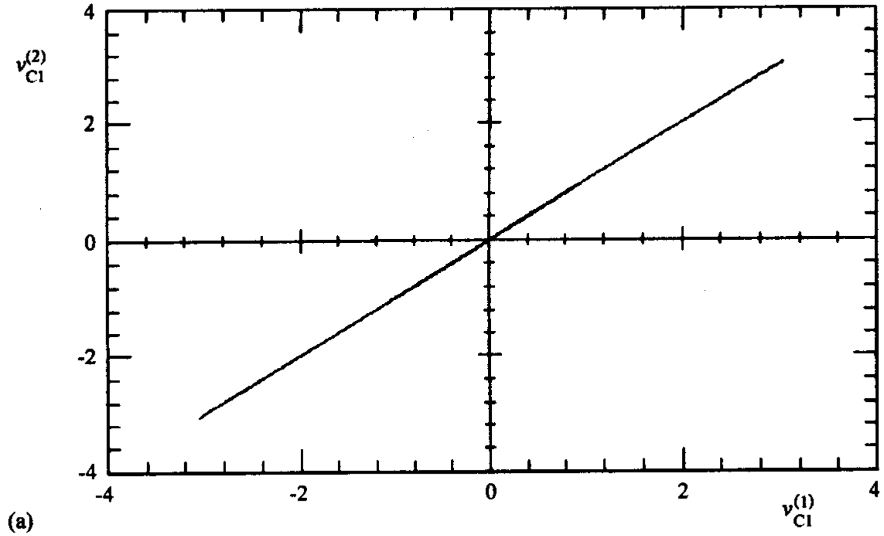
$$\dot{x}' = \alpha(y' - x' - h(x')) \qquad \dot{x}'' = \alpha(y'' - x'' - h(x''))$$
$$\dot{y}' = x' - y' + z' + K(y'' - y') \qquad \dot{y}'' = x'' - y'' + z''$$
$$\dot{z}' = -\beta y' \qquad \dot{z}'' = -\beta y''$$

K is the coupling strength.

Complete synchronization takes place above the critical value $K = K_c \in (1.1, 1.2)$.



$K = 1.1$



$K = 1.2$