CONSENSUS IN NETWORKED MULTI-AGENT SYSTEMS

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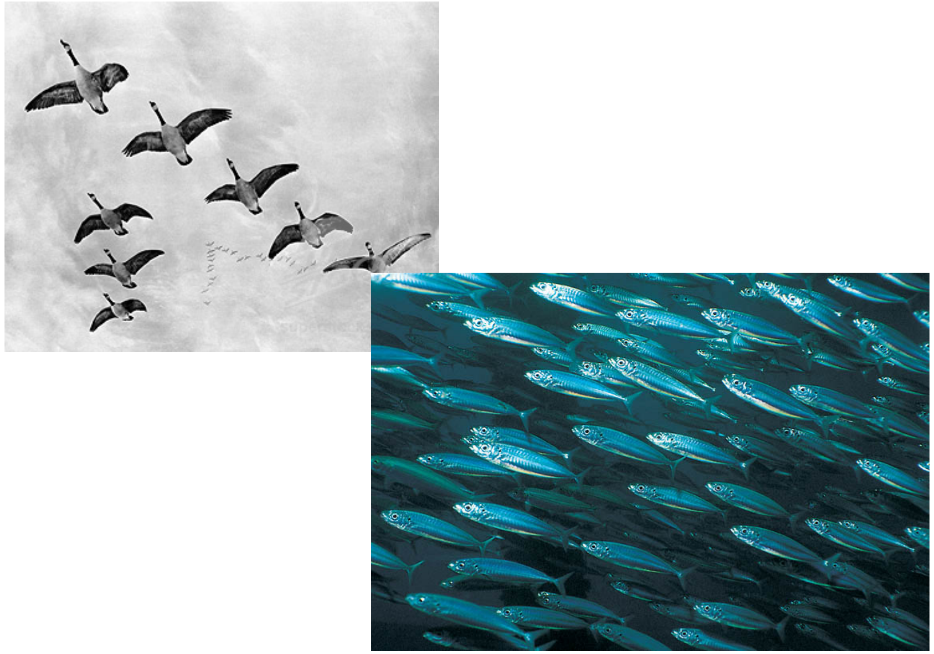
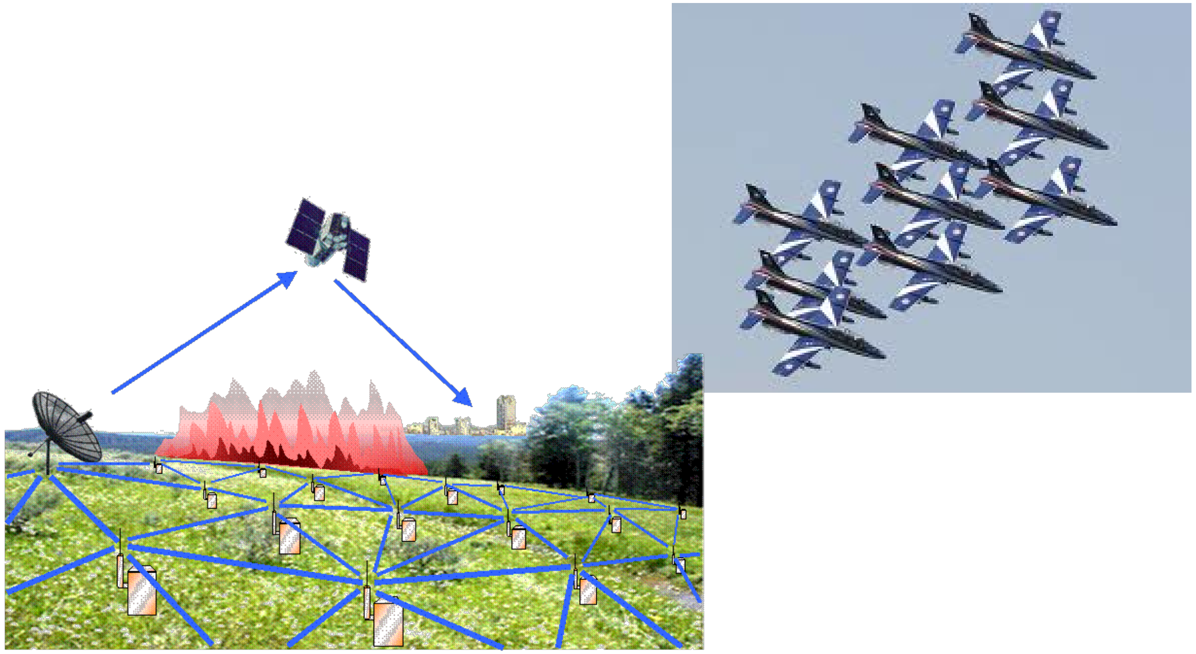
CONSENSUS

Given a set of agents (=dynamical systems):

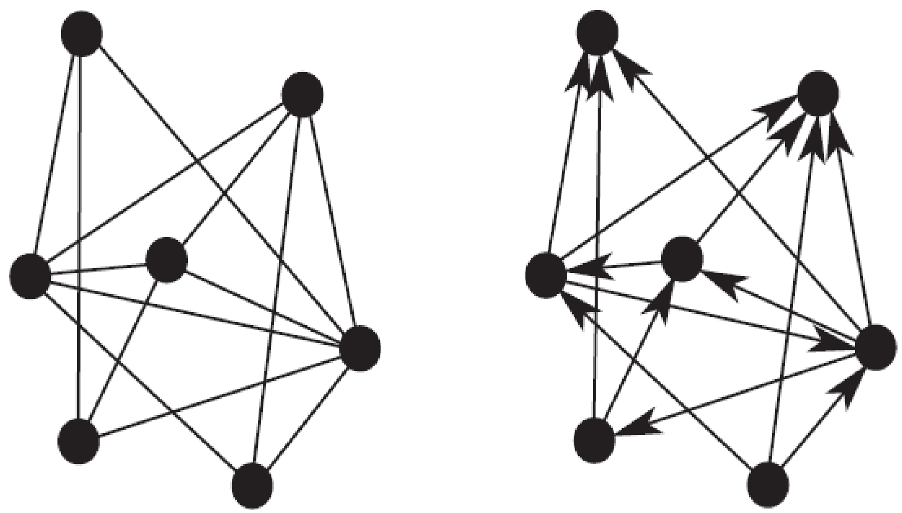
- "consensus" means to reach an agreement regarding a certain quantity of interest, that depends on the state of all agents;
- "consensus algorithm" (or protocol) is the interaction rule that specifies the information exchange between agent.

Analyzing the consensus phenomena... | *...designing the consensus of multi-agent systems*





CONSENSUS ON NETWORKS



Agents are connected through a network:

communication only with neighbours

Each node $i = 1, 2, \dots, N$ hosts a system

$\dot{x}_i(t) = f(x_i(t), u_i(t))$, continuous-time model

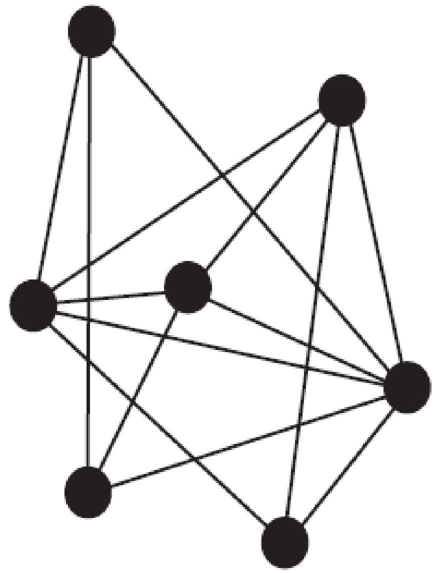
$x_i(t+1) = f(x_i(t), u_i(t))$, discrete-time model

$u_i(t)$ = input = interaction with other systems

(unconstrained) consensus: $x_1 = x_2 = \cdots = x_N$ for $t \to \infty$

Can consensus be reached with partial exchange of information only (=local, distributed, decentralized)?

Consensus in a network of *N* integrator agents



Each node hosts the "simplest" linear system:

$$\dot{x}_i(t) = u_i(t), \quad i = 1,2,\dots,N, \quad x_i \text{ scalar}$$

Undirected, connected network, $V_i = \{j | a_{ij} = 1\}$ = neighbours of i

"diffusive" interaction: proportional to the state unbalance

$$\dot{x}_i(t) = u_i(t) = \sum_{j \in V_i} \left(x_j(t) - x_i(t) \right)$$

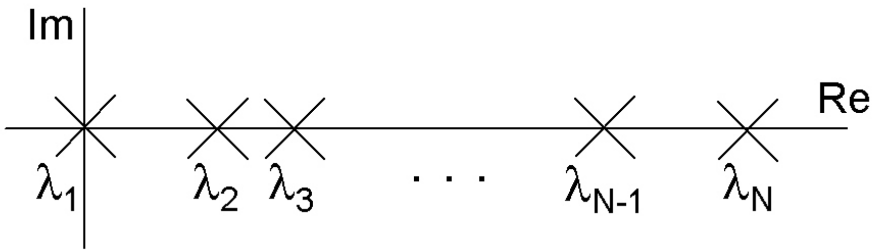
The collective dynamics of $x = [x_1 \quad x_2 \quad \cdots \quad x_N]^{\mathrm{T}}$ are governed by

$$\dot{x} = -Lx$$

where $L = diag(k_1, k_2, \dots, k_N) - A$ is the $N \times N$ Laplacian matrix of the network.

The collective dynamics depend on the graph topology!

Main results:



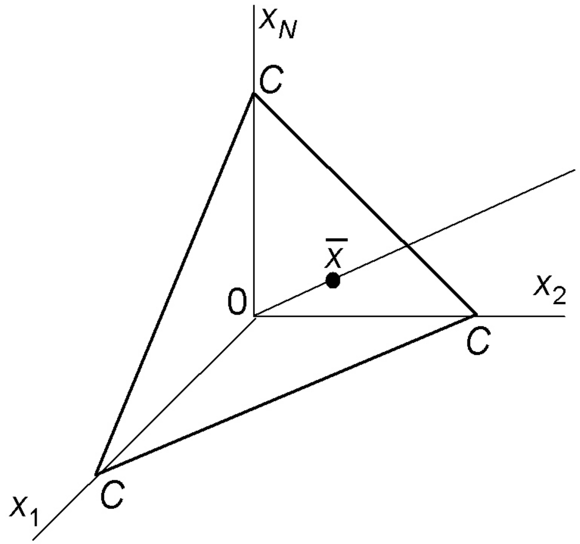
- L is symmetric, irreducible, zero-row-sum, and $(-L)$ is Metzler: eigenvalues are real and $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$

- Possible equilibria are all the (infinitely many) states with $x_1 = x_2 = \cdots = x_N$.

- Since $\sum_i \dot{x}_i = 0$, it turns out that $\sum_i x_i(t) = \sum_i x_i(0) = C$ (constant).

- Thus the only feasible equilibrium $\bar{x}$ has

$$\bar{x}_1 = \bar{x}_2 = \cdots = \bar{x}_N = (1/N)\sum_i x_i(0)$$



- $\lambda_1 = 0$ governs the (trivial) dynamics along the eigenvector $[1 \quad 1 \quad \ldots \quad 1]^{\mathrm{T}}$.

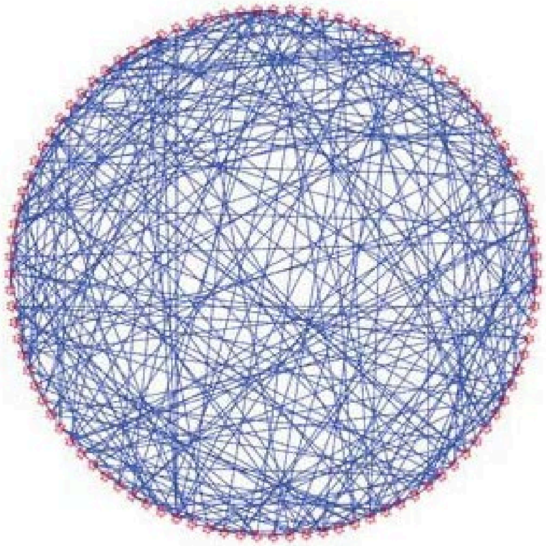
- $-\lambda_N \leq \cdots \leq -\lambda_2 < 0$ govern the dynamics on the plane $\sum_i x_i(t) = C$.

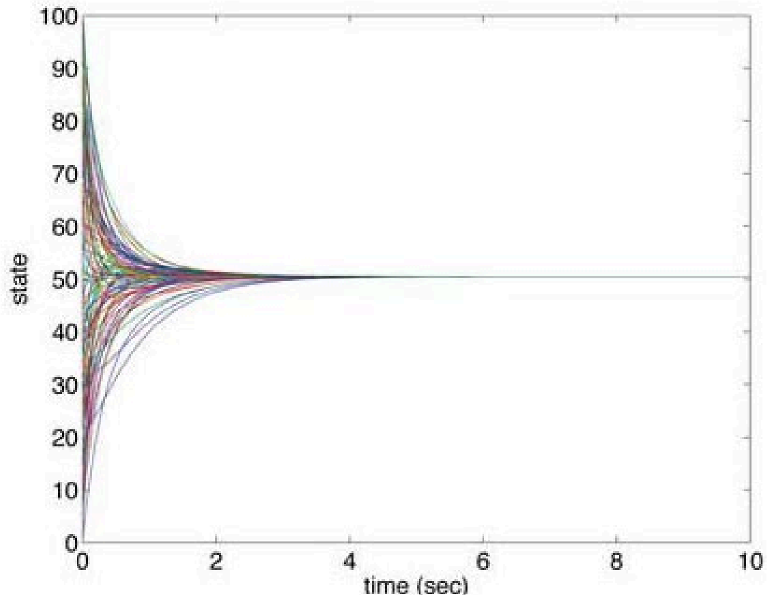
All non-dominant eigenvalues of $(-L)$ are negative $\Rightarrow$ asymptotic convergence to

$$\bar{x} = \left(\frac{1}{N}\sum_i x_i(0)\right)[1 \quad 1 \quad \dots \quad 1]^{\mathrm{T}}$$

- consensus reached: each system converges to the mean value of the initial states

- the convergence speed is dominated by the first sub-dominant eigenvalue, i.e. $x_i(t) - \bar{x}_i \approx \exp(-\lambda_2 t)$: the speed of convergence depends on the topology (the larger λ_2, the faster the convergence).





Network topology and the speed of consensus

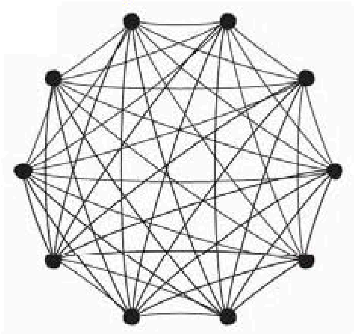
The larger λ_2 ("*algebraic connectivity*"), the faster the convergence.

How does λ_2 depend on the network topology?

the two-node net

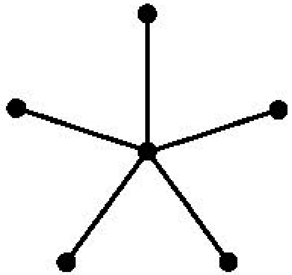
$\lambda_2 = 2$

complete nets



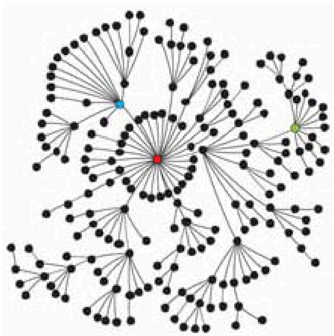
$\lambda_2 = \cdots = \lambda_N = N$

star nets



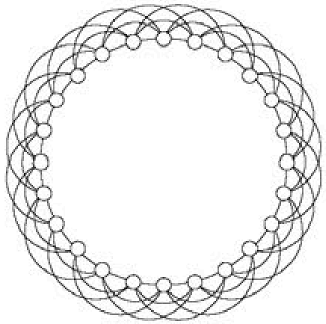
$$\lambda_2 = \cdots = \lambda_{N-1} = 1$$
$$\lambda_N = N$$

scale-free nets



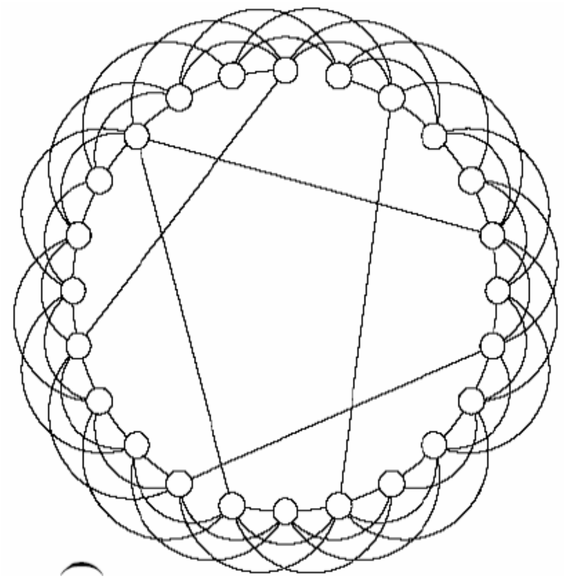
$\lambda_2 \approx 1$

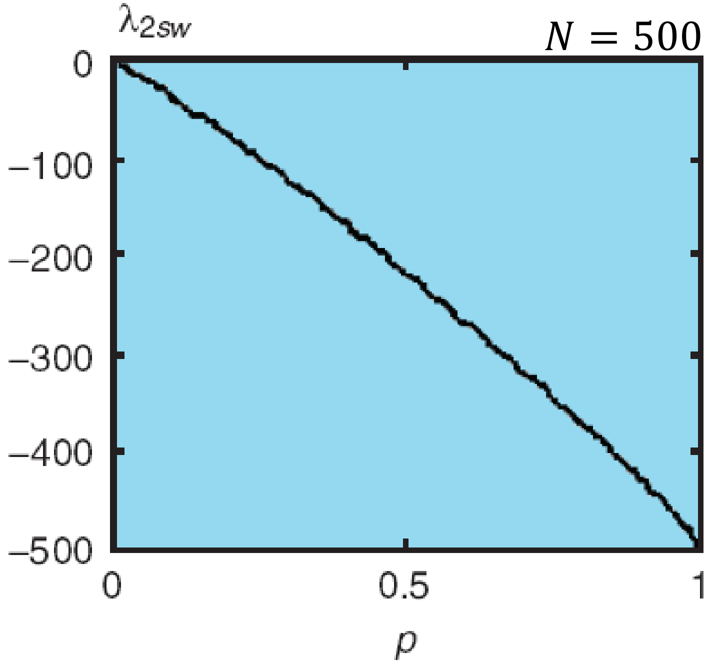
Watt-Strogatz loops



$\lambda_2 \to 0,\ \dfrac{\lambda_N}{\lambda_2} \sim N^2$ as $N \to \infty$

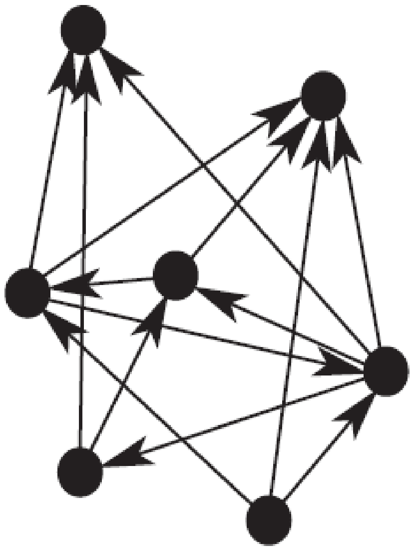
Given a Watts-Strogatz loop, the speed of convergence can be greatly increased by adding "long-distance" connections (small-world network).





Directed networks (Laplacian matrix: $L_{ii} = k_i^{out}$, $L_{ij} = -a_{ij}, i \neq j$):

If the network is strongly connected:



- consensus reached: each system converges to

$$\bar{x} = \alpha[1 \quad 1 \quad \dots \quad 1]^{\mathrm{T}}$$

$\alpha = \sum_i w_i x_i(0)$ and $\sum_i w_i = 1$, thus α is a weighted average of the initial states.

- α depends on the network topology: $[w_1 \quad w_2 \quad \dots \quad w_N]$ is a left eigenvector of the Laplacian L for $\lambda_1 = 0$:

$$[w_1 \quad w_2 \quad \dots \quad w_N]L = 0$$

- If the network is balanced ($k_i^{in} = k_i^{out}$ for all i), then

$$\alpha = \left(\frac{1}{N}\sum_i x_i(0)\right)\text{: convergence to the average initial state}$$

- The speed of convergence is dominated by the first sub-dominant eigenvalue of $L_S = (L + L^{\mathrm{T}})/2$.

Possible extensions:

- different forms of the coupling function $u_i = h(x)$
- network design for optimizing consensus performance
- other (less trivial) agent models (linear or nonlinear)
- time-varying network topology (e.g., mobile agents)
- ...
- consensus on oscillatory behaviour: synchronization